a short lecture on

Waveform Inversion

by

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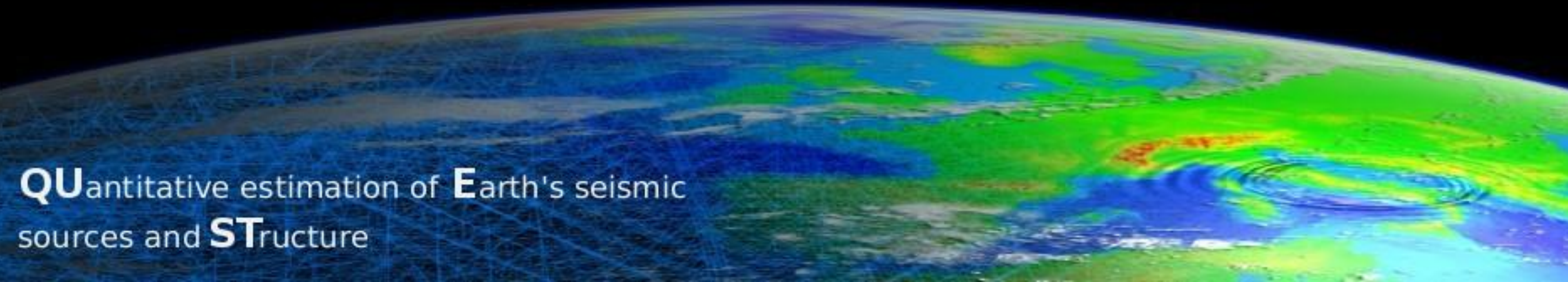
Department of Earth Sciences



QUantitative estimation of **E**arth's seismic sources and **ST**ructure

Solutions to numerous societal and academic problems

rely on increasingly detailed

images of the Earth‘s interior sturcture:

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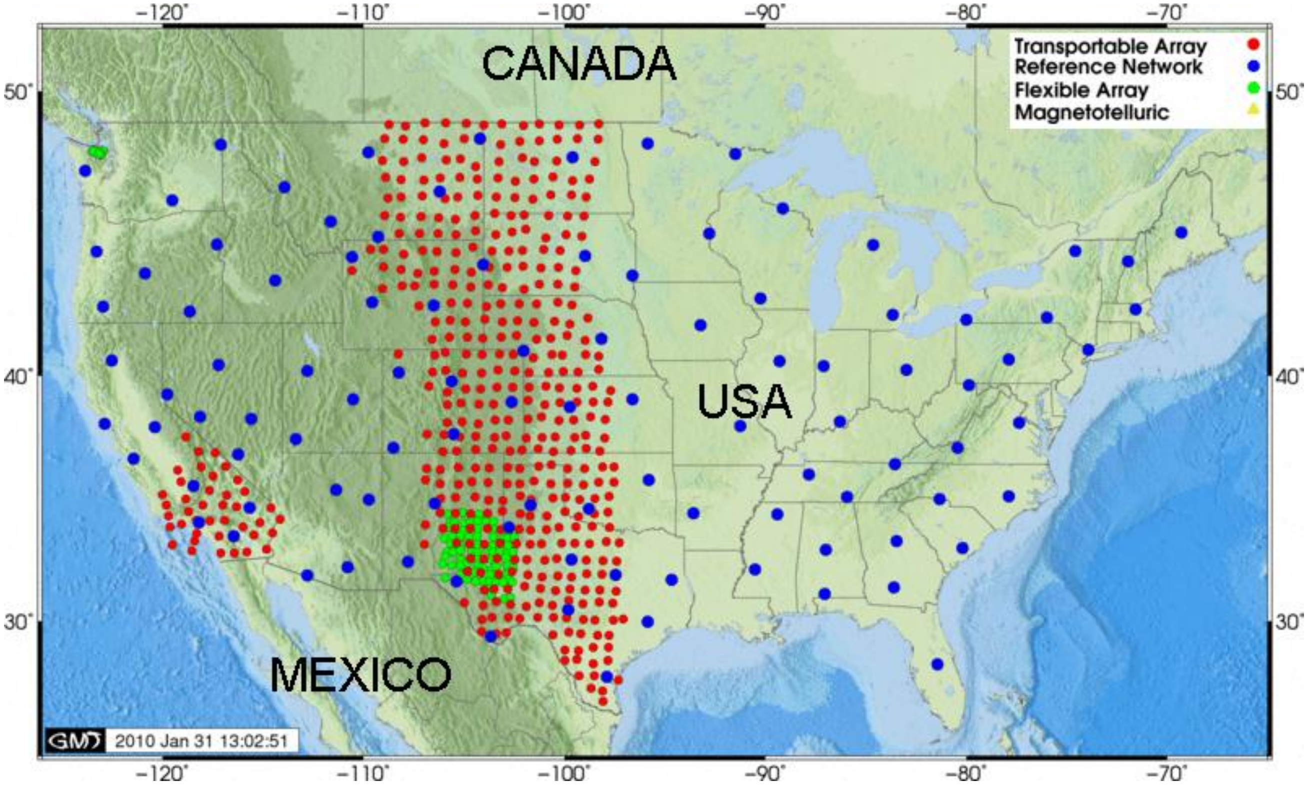
1) Monitoring of the CNTBT
2) Reliable tsunami warnings (see end of this talk)
3) Prediction of strong seismic ground motion
4) Exploration for natural resources
5) Nature of plumes in the deep mantle (see poster by Florian Rickers)
6) Composition and dynamics of the solid Earth

Information on the structure of the Earth is limited by

the uneven distribution of receivers:

- oceans
- mountain ranges
- deserts including Antarctica
- politics and financial resources



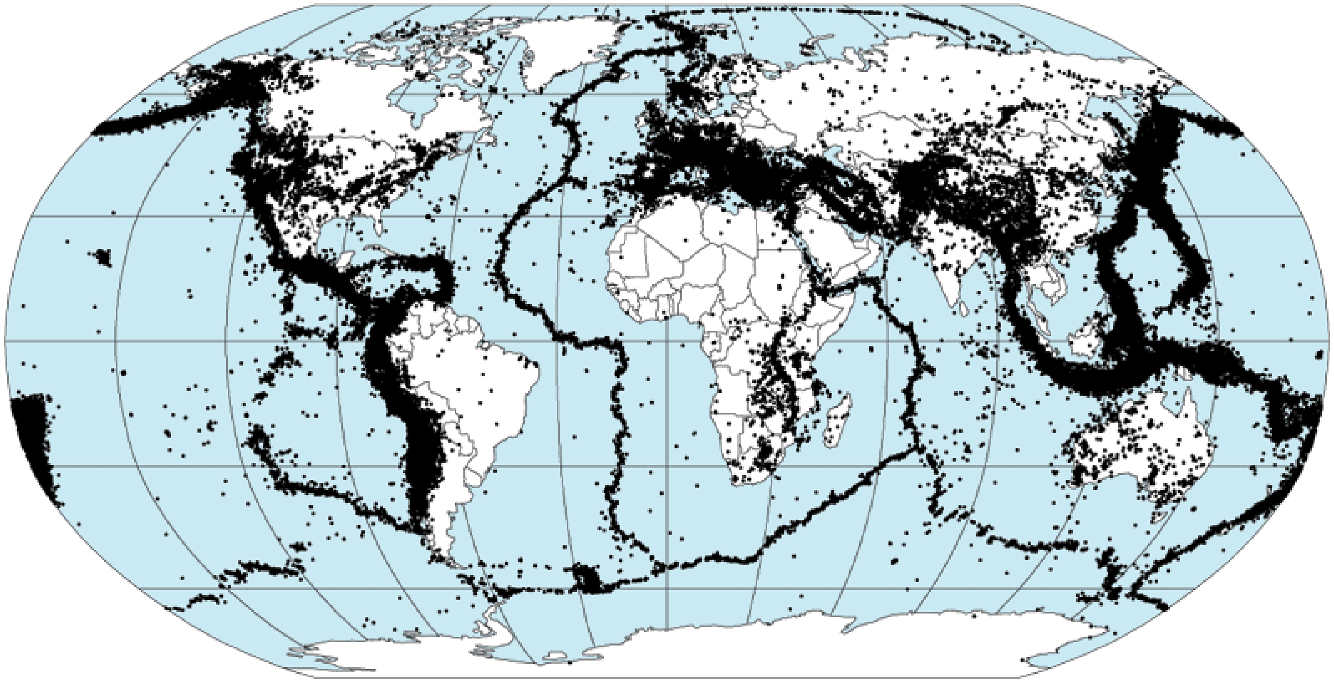
seismic station coverage

North America

Information on the structure of the Earth is limited by

the uneven distribution of sources:

- earthquakes in a few tectonically active areas (passive imaging)
- financial resources (active imaging)
- environmental issues (active imaging)



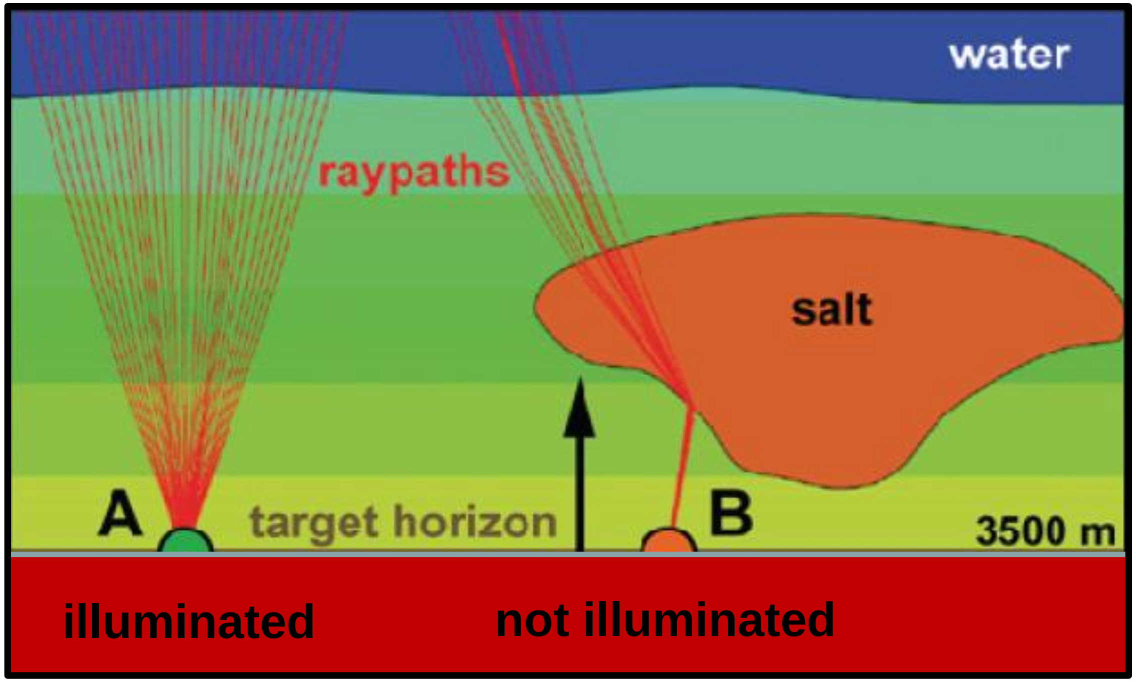
Epicenters 1963 - 1998

358,214 Events

Information on the structure of the Earth is limited by

the physics of seismic wave propagation:

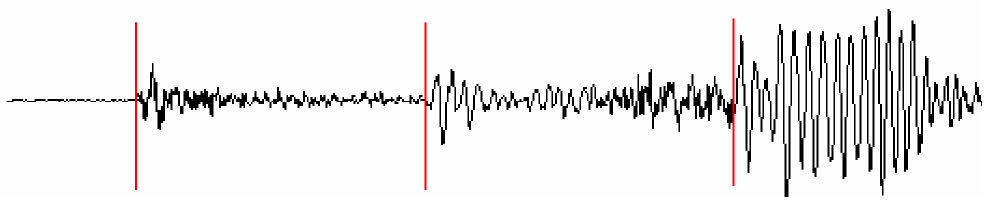
- attenuation prevents propagation of high-frequency waves
- irregular sampling due to the presence heterogeneity



Lecomte et al., 2009

Underside of the salt body is not illuminated due to heterogeneity.

Want learn as much as possible about the Earth‘s properties?



Exploit as much waveform information as possible!

STEPS TO BE TAKEN

1. Solution of the forward problem

Computation of accurate synthetic seismograms for heterogeneous Earth models

Numerical methods to solve the seismic wave equation

2. Quantification of waveform misfit

Exploit as much information as possible while conforming to the physics of the problem

3. Misfit minimisation: Gradient methods

Iterative reduction of the misfit using gradient methods

Convergence: initial model and multi-scale approach

4. Efficient computation of the gradient: The adjoint method

Adjoint wave field and time reversal

Sensitivity kernels

OUTLINE

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Adjoint wave field and time reversal

Sensitivity kernels

5. Applications

6. Challenges and directions of future research

1. Forward Problem Solution

Forward problem: The seismic wave equation

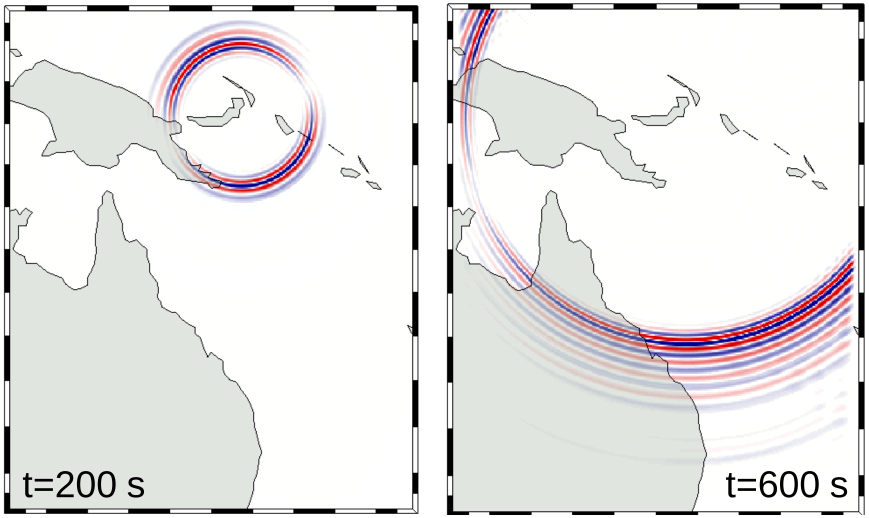
density

elastic tensor

force density

$$\rho\ \ddot{u}_i - \frac{\partial}{\partial x_j}\left(C_{ijkl} * \frac{\partial}{\partial x_k} u_l \right) = f_i$$

elastic displacement field, **u**



wave field @ 100 km depth

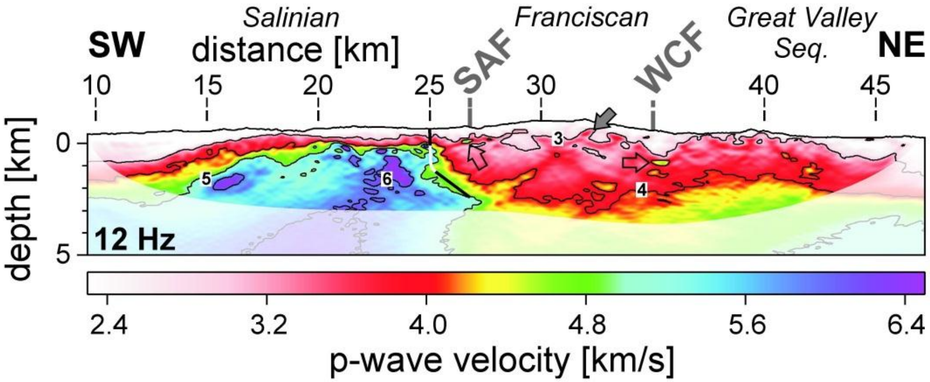
Forward problem: The seismic wave equation

density elastic tensor force density

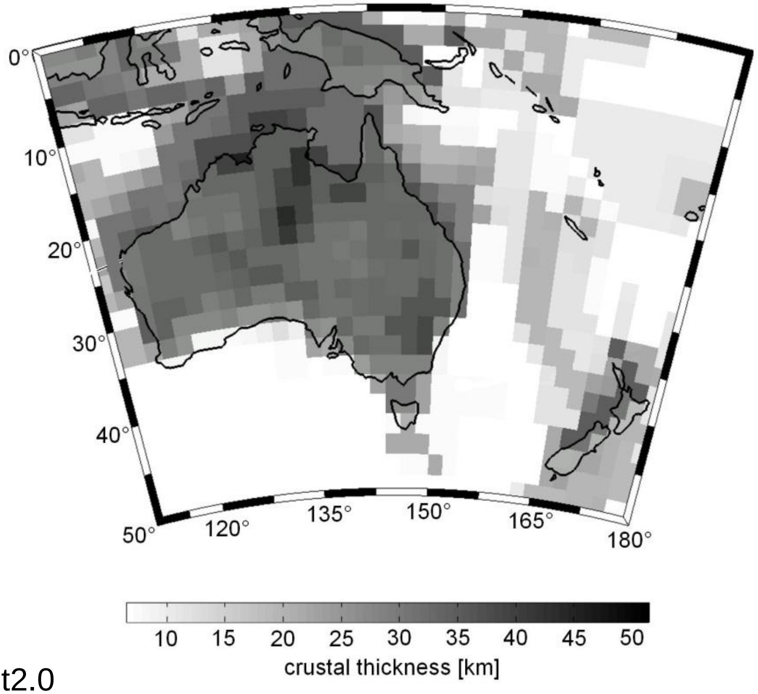
$$\rho\, \ddot{u}_i - \frac{\partial}{\partial x_j}\left(C_{ijkl} * \frac{\partial}{\partial x_k} u_l \right) = f_i$$

elastic displacement field, **u**

<u>No</u> exact solutions exist for heterogeneous media!



Bleibinhaus et al. (2007)



Crust2.0

Forward problem: The seismic wave equation

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$$\rho\,\ddot{u}_i - \frac{\partial}{\partial x_j}\left(C_{ijkl} * \frac{\partial}{\partial x_k} u_l\right) = f_i$$

elastic displacement field, **u**

No exact solutions exist for heterogeneous media!

Find numerical solutions

that allow for

nearly arbitrary heterogeneities.

Forward problem: Discretisation of spatial derivatives

$$\rho\ \ddot{u}_i - \frac{\partial}{\partial x_j}\left(C_{ijkl} * \frac{\partial}{\partial x_k} u_l \right) = f_i$$

spatial discretisation

- **replace derivatives by finite-difference approximations**

  finite-difference method

- **approximate u(x,t) by polynomials**

  discontinuous Galerkin method, spectral-element method

- **approximate derivatives in the wave number domain**

  pseudo-spectral methods

- **…**

Forward problem: Discretisation of spatial derivatives

$$\rho\, \ddot{u}_i - \frac{\partial}{\partial x_j}\left(C_{ijkl} * \frac{\partial}{\partial x_k} u_l \right) = f_i$$

spatial discretisation

semi-discrete wave equation:

$$\overbrace{\mathbf{M}}^{\text{mass matrix}} \cdot \underbrace{\ddot{\bar{\mathbf{u}}}(t)} + \overbrace{\mathbf{K}}^{\text{stiffness matrix}} \cdot \bar{\mathbf{u}}(t) = \bar{\mathbf{f}}(t)$$

discrete approximation of the displacement field

- wave field sampled at a finite number of grid points (finite-difference method)
- polynomial coefficients (spectral-element & discontinuous Galerkin methods)

Forward problem: Discretisation of spatial derivatives

semi-discrete wave equation: $\mathbf{M} \cdot \ddot{\bar{\mathbf{u}}}(t) + \mathbf{K} \cdot \bar{\mathbf{u}}(t) = \bar{\mathbf{f}}(t)$

solution either in

<ins>the time domain</ins> or <ins>the frequency domain</ins>

easy to invert (diagonal)

$$\ddot{\bar{\mathbf{u}}}(t) = \mathbf{M}^{-1} \cdot [\bar{\mathbf{f}}(t) - \mathbf{K} \cdot \bar{\mathbf{u}}(t)]$$

- discretise the time derivative
- iteratively advance from *t* to *t*+Δ*t* :

$$\bar{u}(t) \rightarrow \bar{u}(t+\Delta t)$$

- no need to invert **K**

- **time-domain waveform inversion**
- preferred in large-scale 3D applications

$$-\omega^2 \mathbf{M} \cdot \bar{\mathbf{u}}(\omega) + \mathbf{K} \cdot \bar{\mathbf{u}}(\omega) = \bar{\mathbf{f}}(\omega)$$

- define impedance matrix:

$$\mathbf{L}(\omega) := -\omega^2 \mathbf{M} + \mathbf{K}$$

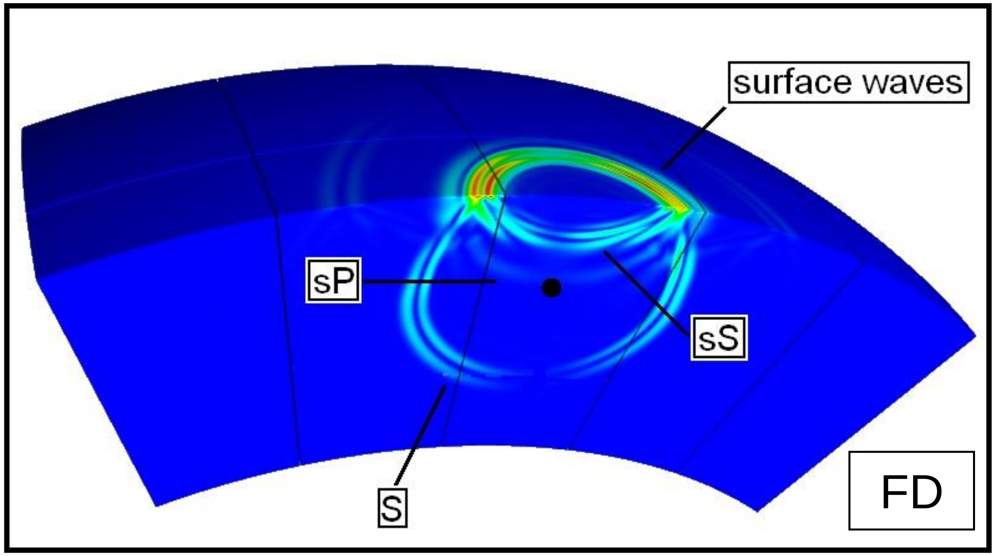
- solve the linear system:

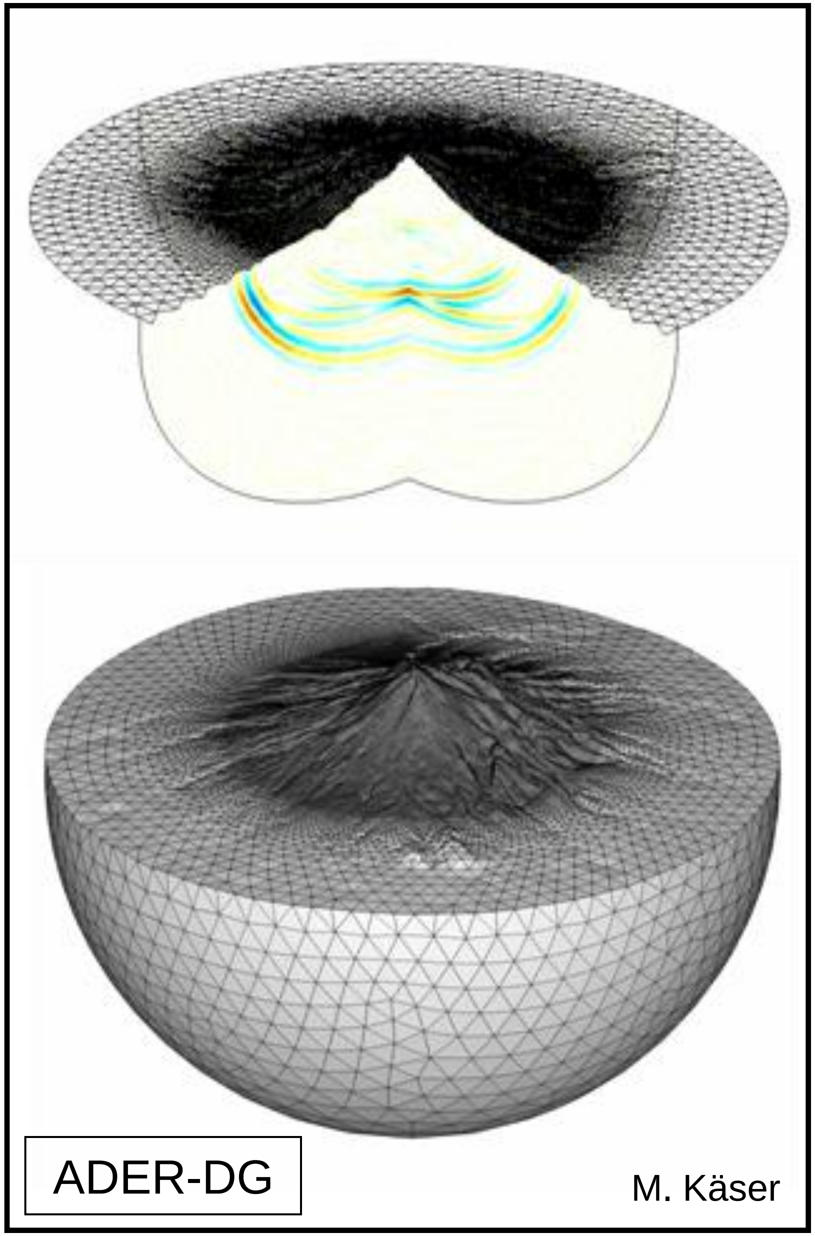
$$\bar{\mathbf{u}}(\omega) = \mathbf{L}^{-1}(\omega) \cdot \bar{\mathbf{f}}(\omega)$$

- **frequency-domain waveform inversion**
- efficient for many different sources
- preferred strategy in 2D

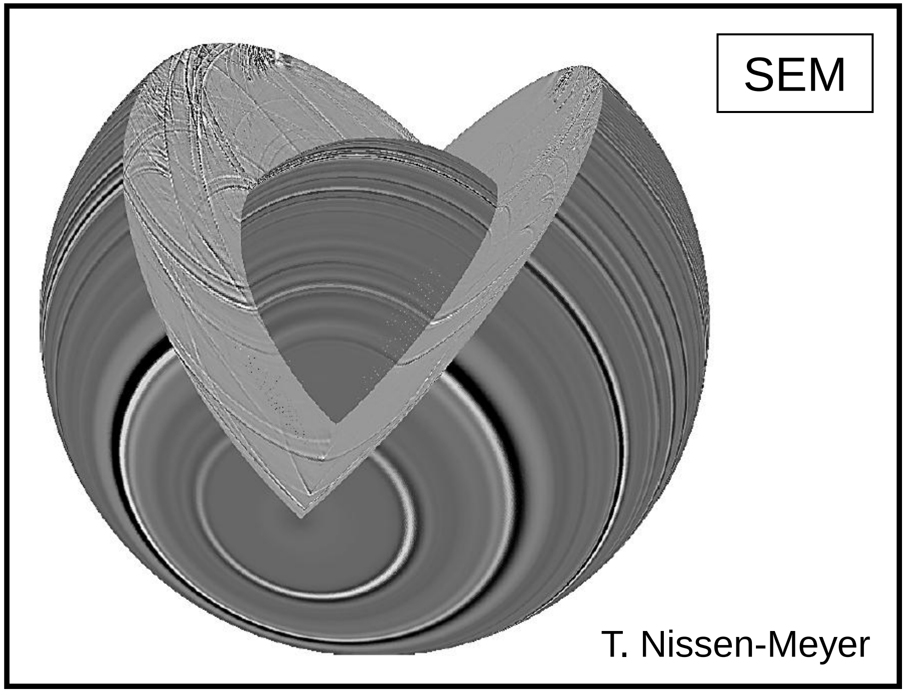
see the talks by Jean Virieux

Forward problem: Examples





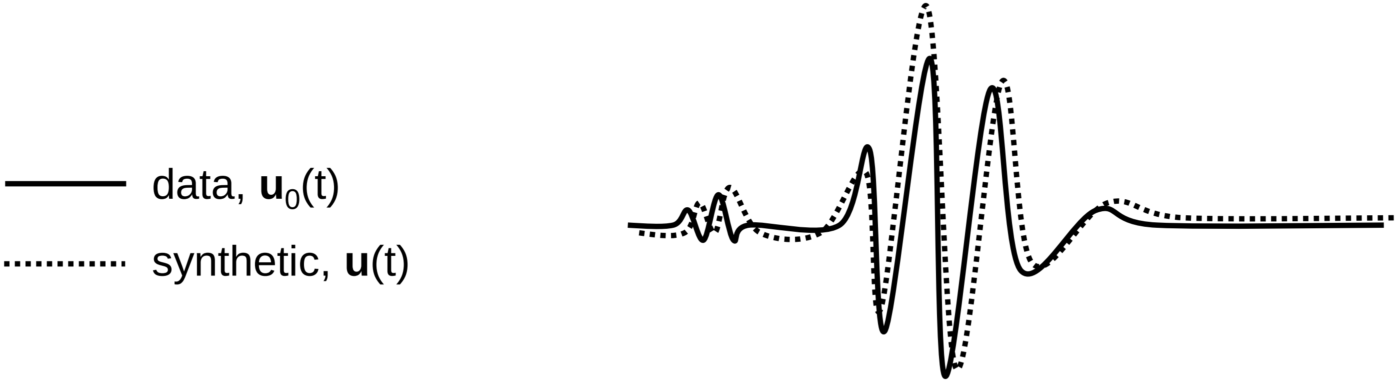
M. Käser



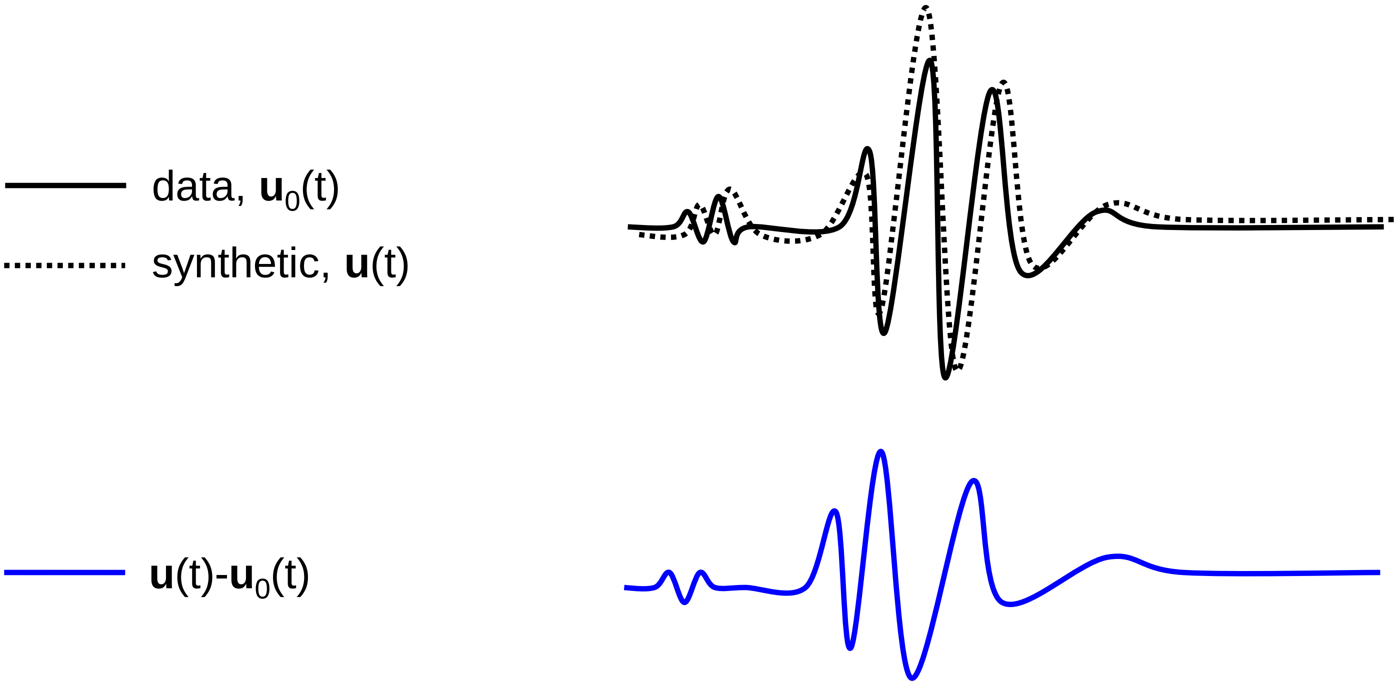
T. Nissen-Meyer

2. Quantification of waveform misfit

Misfit quantification: The L_2 norm

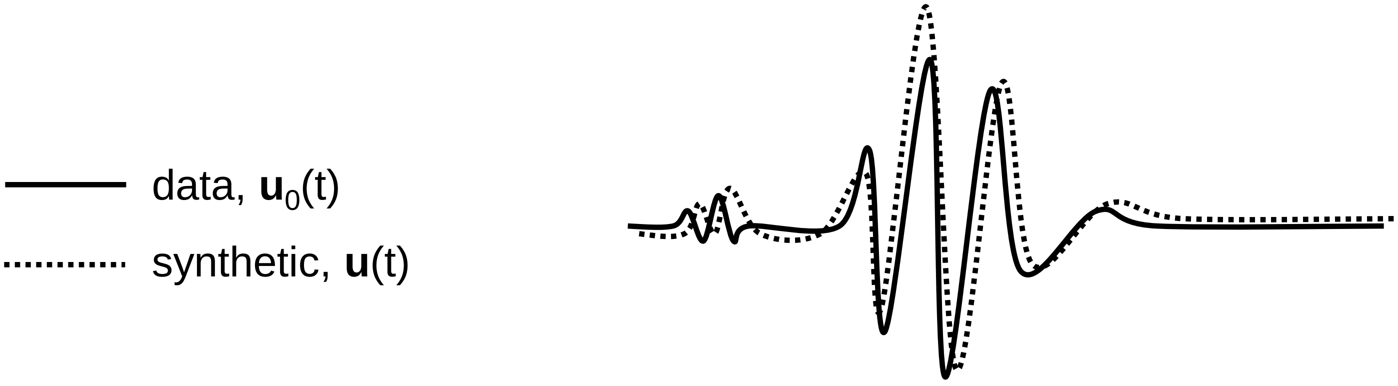


Misfit quantification: The L_2 norm



L_2 waveform misfit: $\chi = \sqrt{\int_t \left[u(t) - u_0(t)\right]^2 dt}$

Misfit quantification: The L_2 norm



L_2 waveform misfit:

$$\chi = \sqrt{\int_t [u(t) - u_0(t)]^2 dt}$$

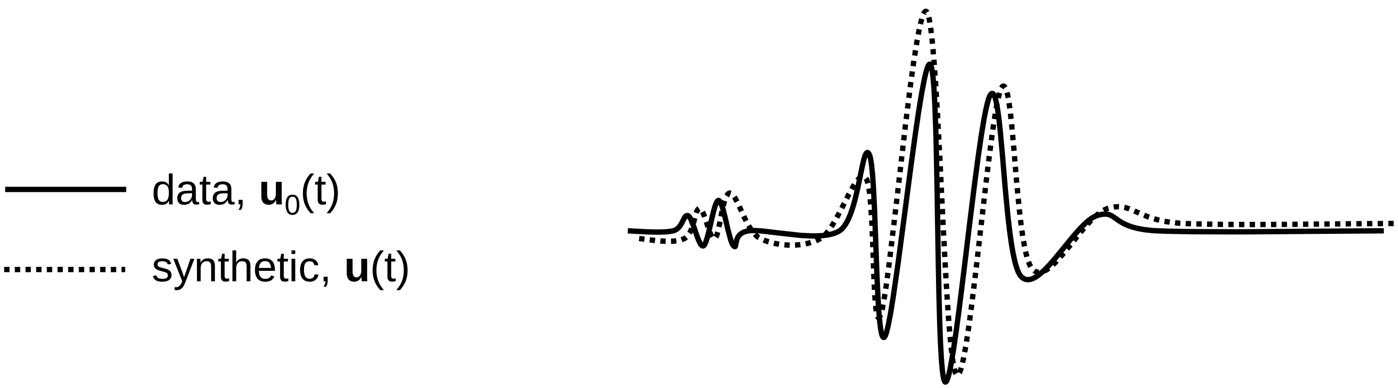
advantages

- easy and fast to implement
- uses the complete waveform

disadvantages

- not robust
- very nonlinearly related to long-wavelength structure
- over-emphasises large-amplitde waves

Misfit quantification: The L_2 norm



L_2 waveform misfit:

$$\chi = \sqrt{\int_t [u(t) - u_0(t)]^2 \, dt}$$

advantages

- easy and fast to implement
- uses the complete waveform

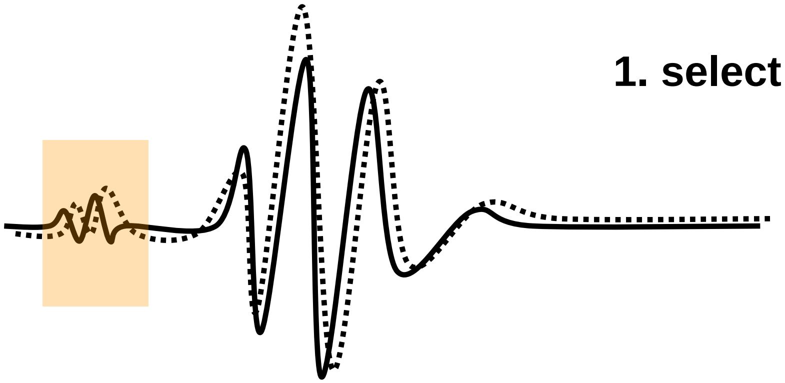
disadvantages

- not robust
- very nonlinearly related to long-wavelength structure
- over-emphasises large-amplitde waves

Not well suited for realistic applications!

Time-like measures of waveform misfit.

Misfit quantification: Cross-correlation time shifts

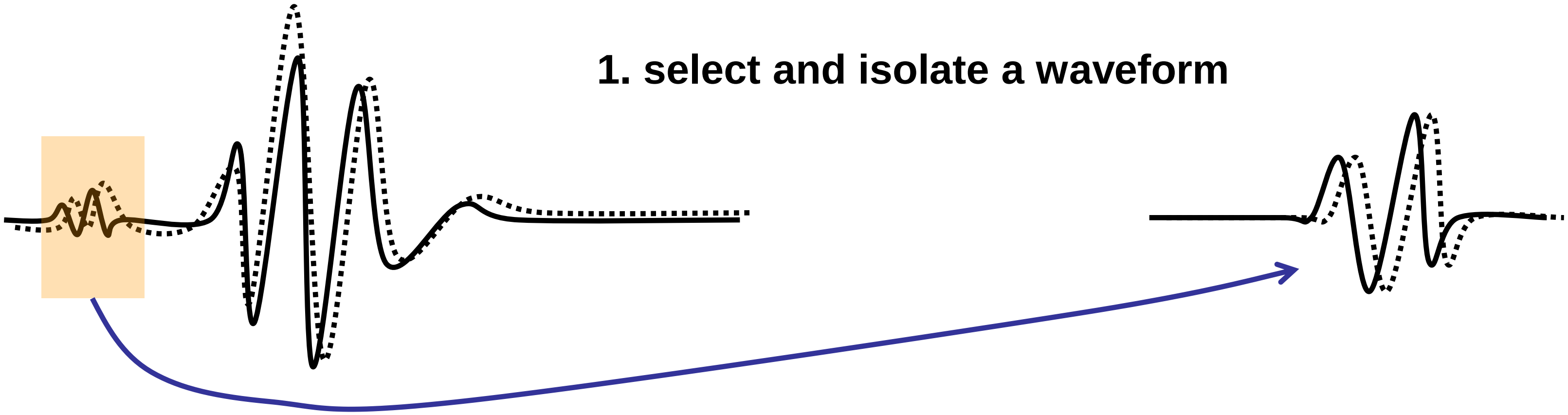


1. select and isolate a waveform

Luo & Schuster, 1991

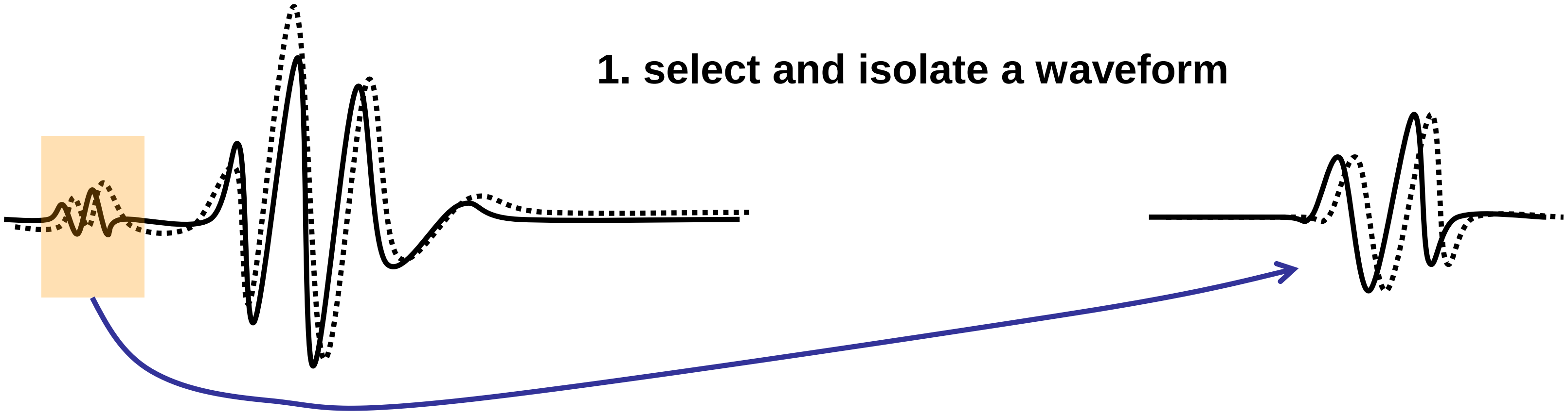
Used before in surface wave analysis.

Misfit quantification: Cross-correlation time shifts



1. select and isolate a waveform

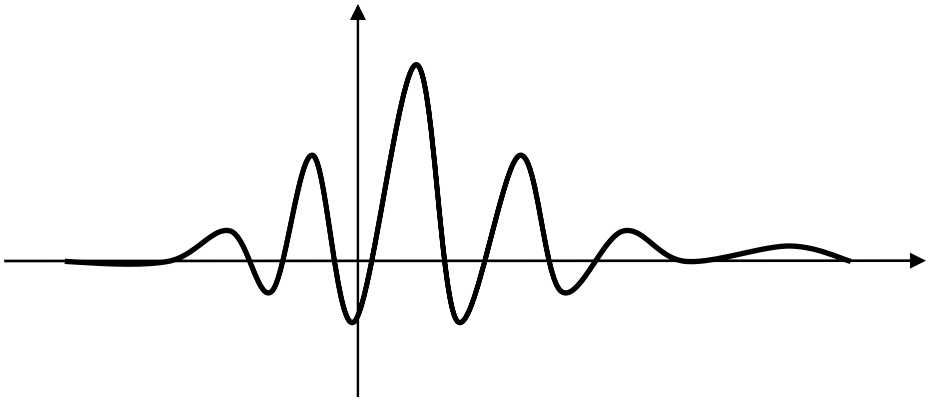
Misfit quantification: Cross-correlation time shifts



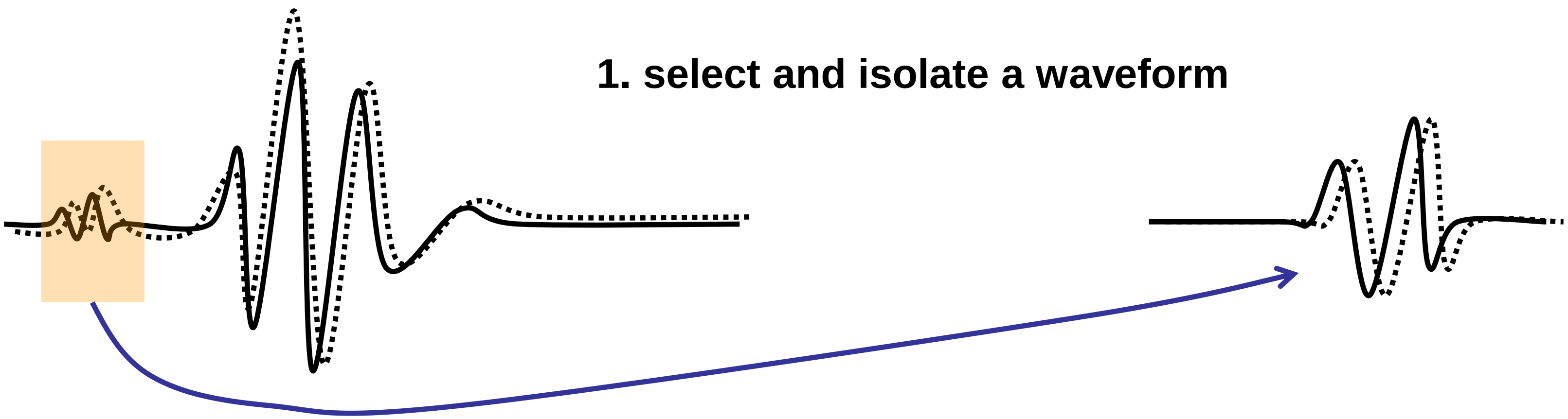
1. select and isolate a waveform

2. compute correlation function

$$C(t) = \int_{\tau} u(t+\tau)\, u_0(\tau)\, d\tau$$



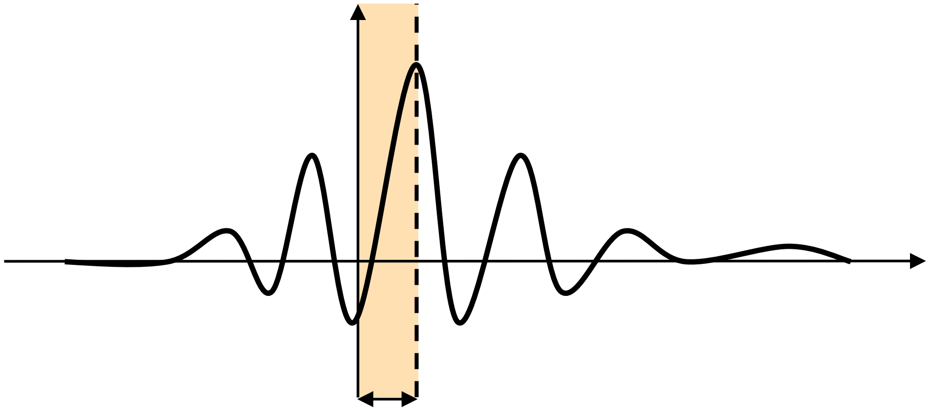
Misfit quantification: Cross-correlation time shifts



1. select and isolate a waveform

2. compute correlation function

$$\chi = \Delta T^2$$

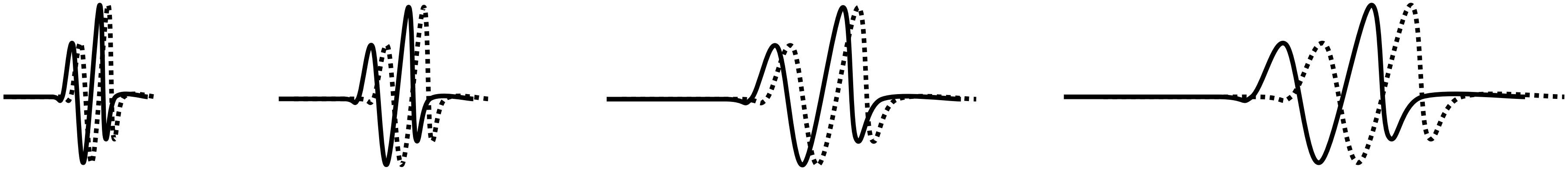


ΔT = **cross-correlation time shift**

Misfit quantification: Cross-correlation time shifts

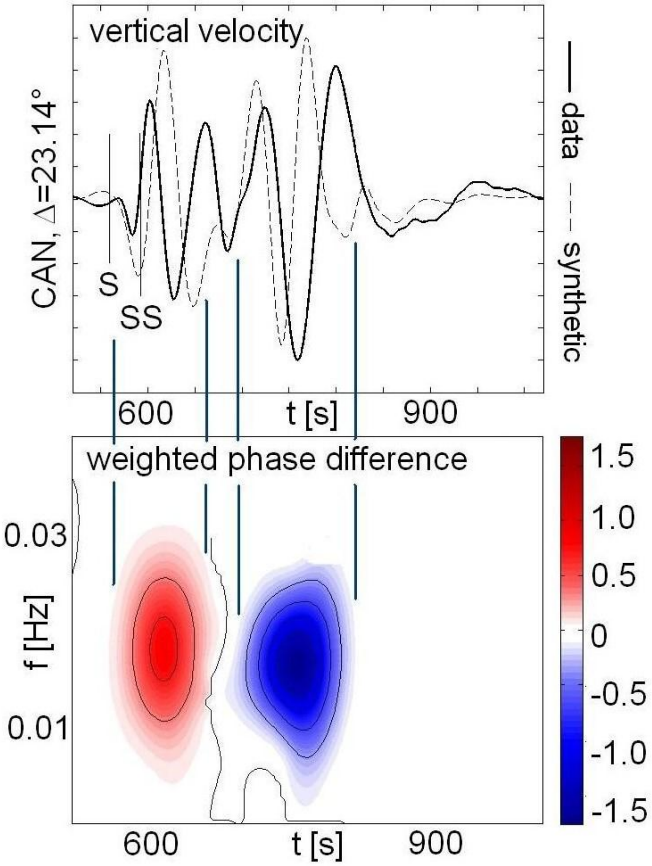
Refinement of the cross-correlation technique:

measurments in multiple frequency bands



➡ **Extract more information about the structure of the Earth.**

ask Karin Sigloch and Guust Nolet for more details.

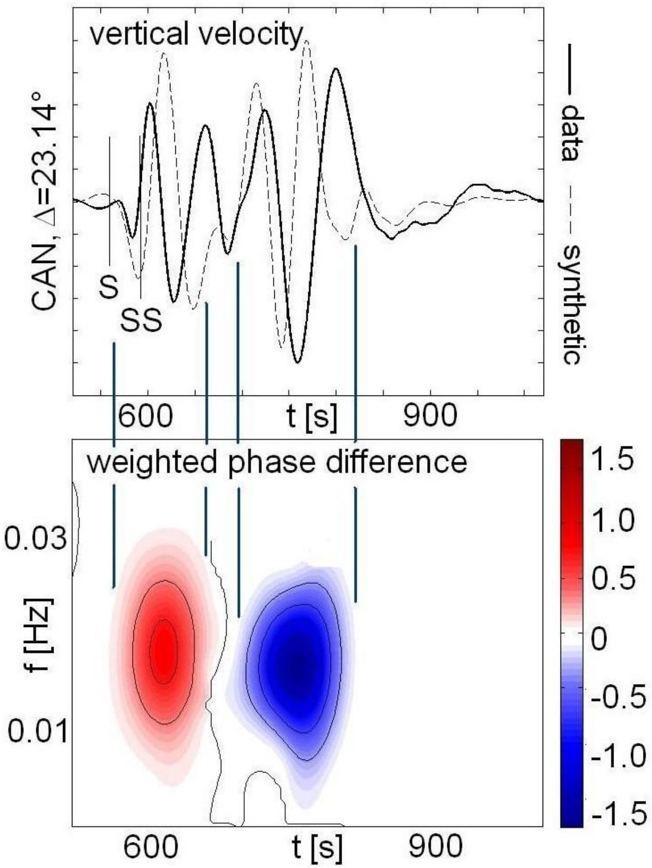


Time-frequency misfits

phase differences as functions of time and frequency

inspired by Kristekova et al., BSSA 2006

Misfit quantification: Time-frequency misfits



Time-frequency misfits

phase differences as functions of time and frequency

- **quasi-linearly related to Earth structure**

  improves convergence

- **independent of amplitudes**

  reliably measurable, deep structure information

- **applicable to complex waveforms**

  interfering waves, unidentifyable waves

- **continuous in frequency**

  no discrete frequency bands

Fichtner et al., GJI 2008

Misfit quantification

The design of suitable misfit measures a major challenge !

Instantaneous phase: Bozdag & Trampert, GJI (2010), F. Rickers‘ poster

Robust measures: Crase et al., Geophysics (1990), Brossier et al., GRL (2009)

3. Misfit minimisation

Misfit minimisation: **Gradient methods**

1. Start from initial Earth model $\mathbf{m}_0$

2. Update according to $\mathbf{m}_{i+1} = \mathbf{m}_i + \underbrace{\gamma_i}_{\text{step length}} \underbrace{\mathbf{h}_i}_{\text{descent direction}} \,,\quad \text{with} \quad \chi(\mathbf{m}_{i+1}) < \chi(\mathbf{m}_i)$

Misfit minimisation: Gradient methods

1. Start from initial Earth model $\mathbf{m}_0$

2. Update according to $\mathbf{m}_{i+1} = \mathbf{m}_i + \underset{\substack{\uparrow \\ \text{step length}}}{\gamma_i}\, \underset{\substack{\uparrow \\ \text{descent direction}}}{\mathbf{h}_i}\,,$ with $\chi(\mathbf{m}_{i+1}) < \chi(\mathbf{m}_i)$

$$\boxed{h_i \propto -\frac{\partial \chi}{\partial m}}$$

<u>The family of gradient methods:</u>

- **method of steepst descent:** $h_i = -\partial\chi / \partial m$
- **conjugate-gradient methods**
- **Newton and Newton-like methods**
- **variable-metric methods**
- **…**

Misfit minimisation: Gradient methods

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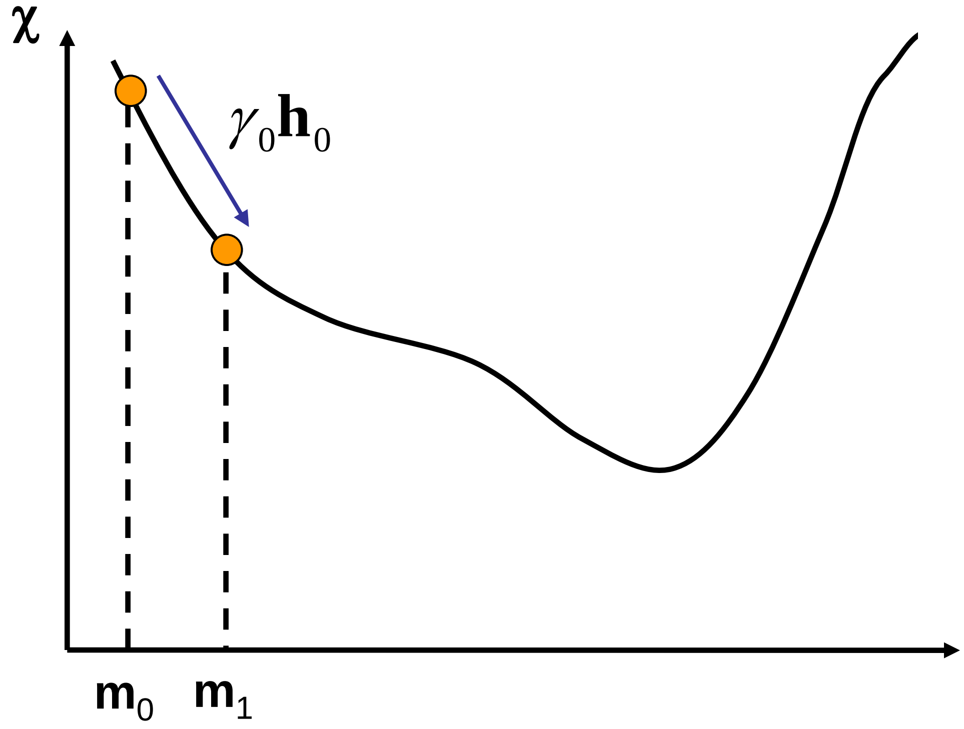
step length ⟶ γ_i

descent direction ⟶ $\mathbf{h}_i$



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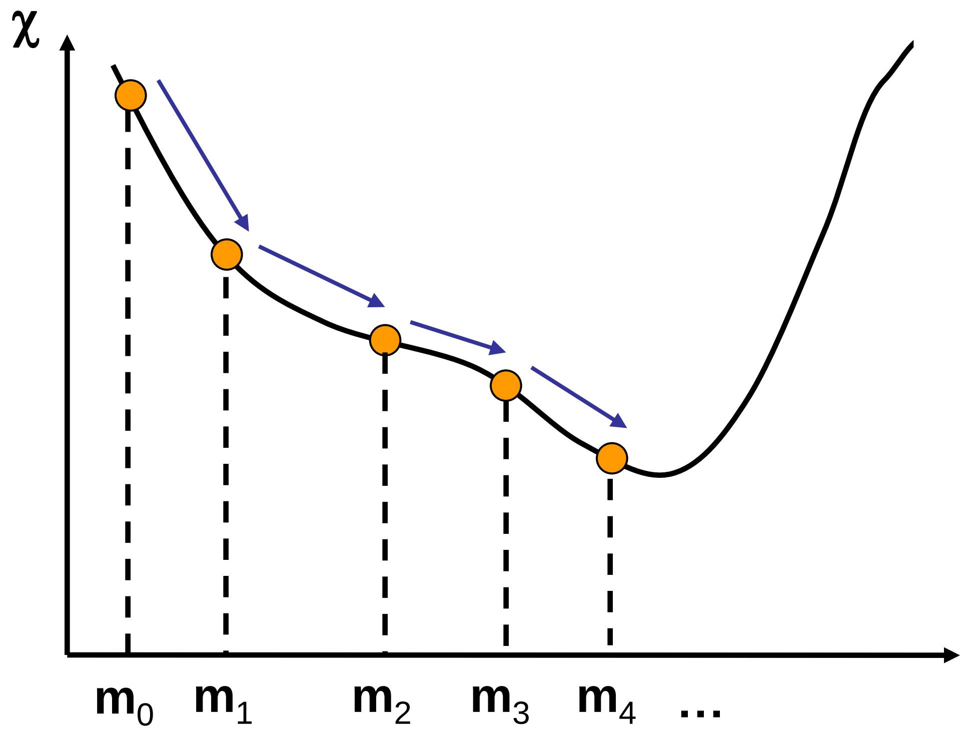
step length → γ_i; descent direction → $\mathbf{h}_i$



Iteratively approach the minimum misfit by following the local descent directions.

Misfit minimisation: Importance of the initial model

1. Start from initial Earth model $\mathbf{m}_0$

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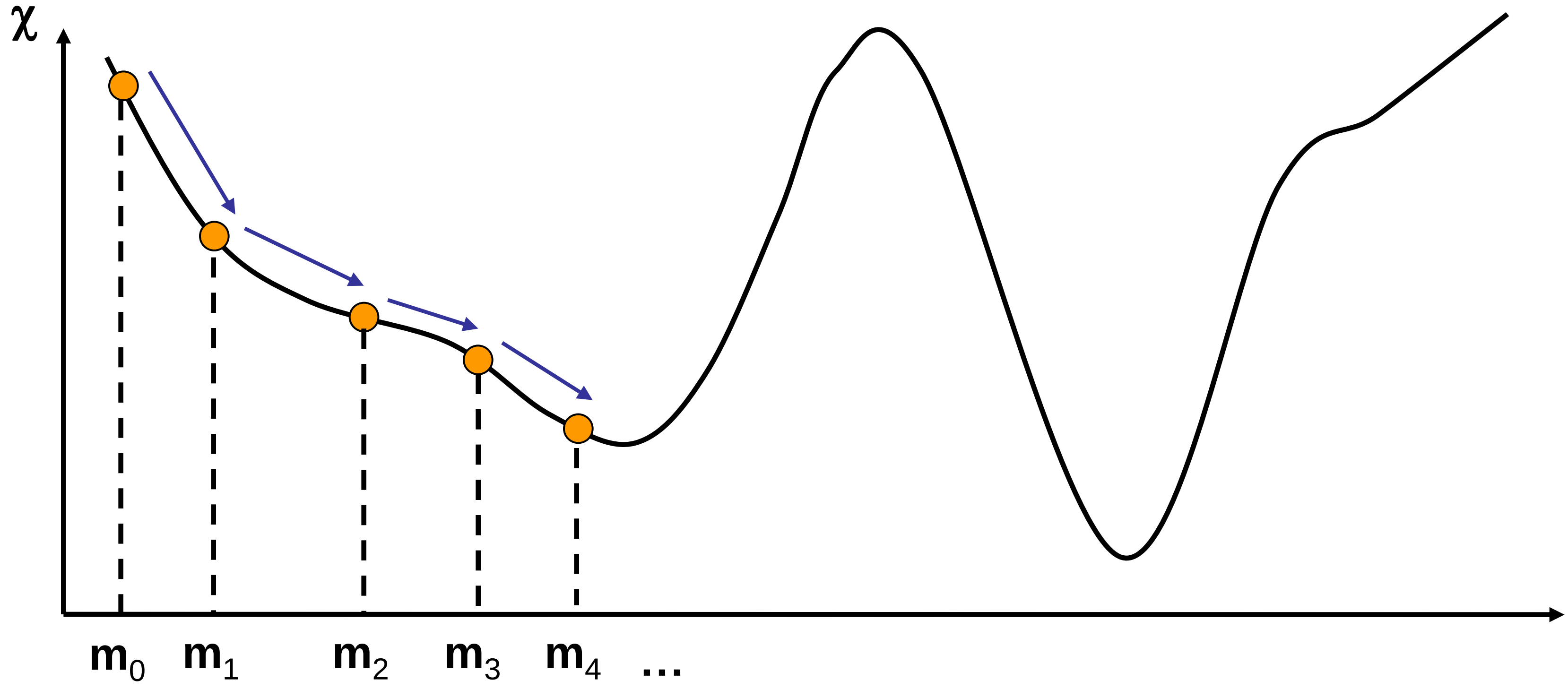
step length

descent direction



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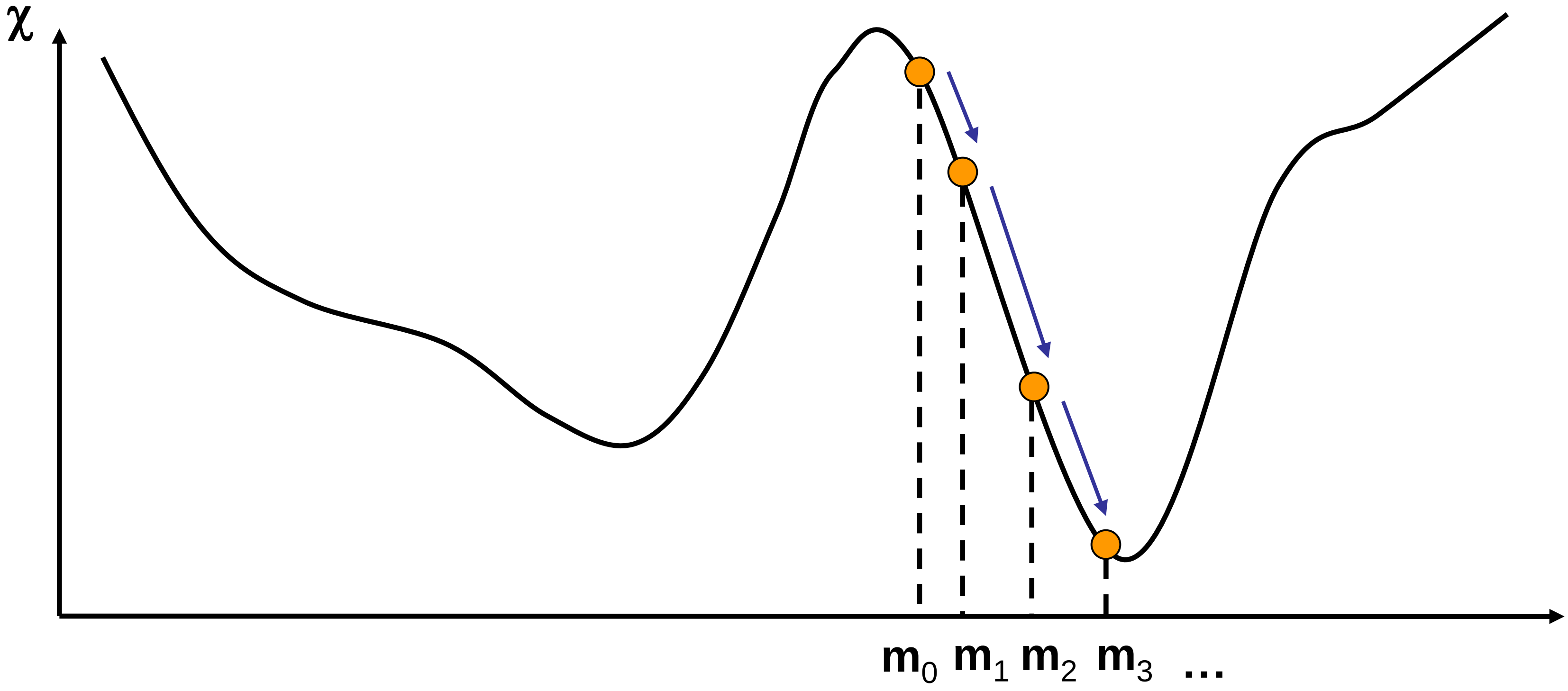
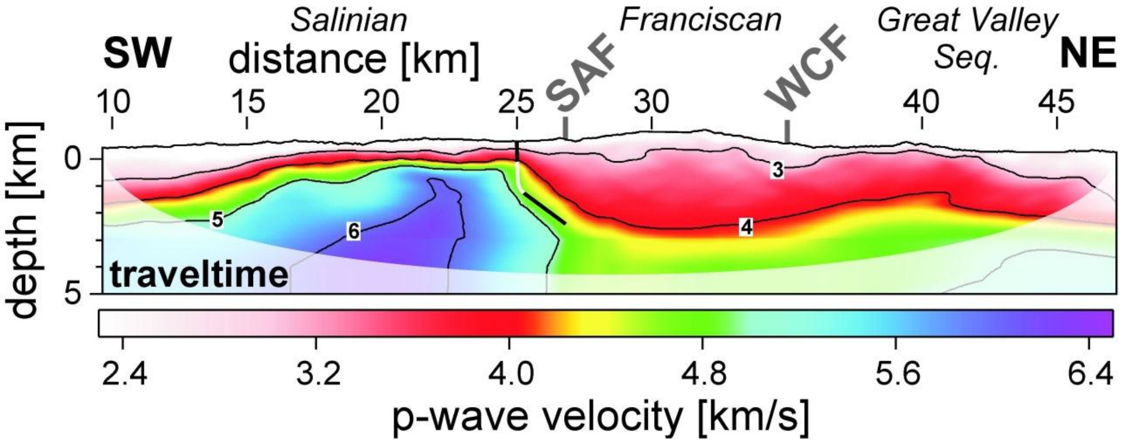


Misfit minimisation: Importance of the initial model

- Gradient methods are local.
- Convergence to the global minimum relies on a good initial model.
- Good initial model: e.g. long-wavelength model from ray tomography.



Bleibinhaus et al., 2007

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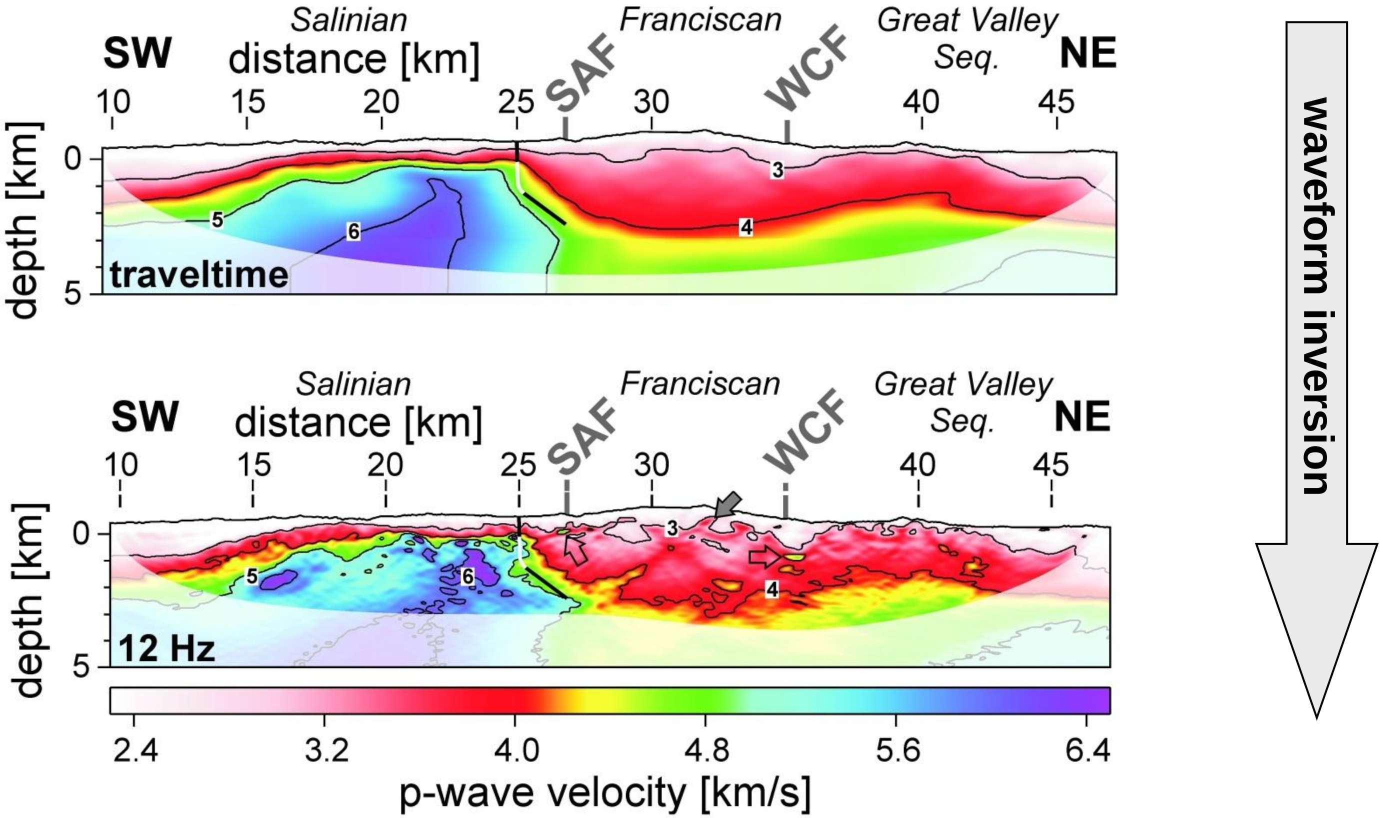
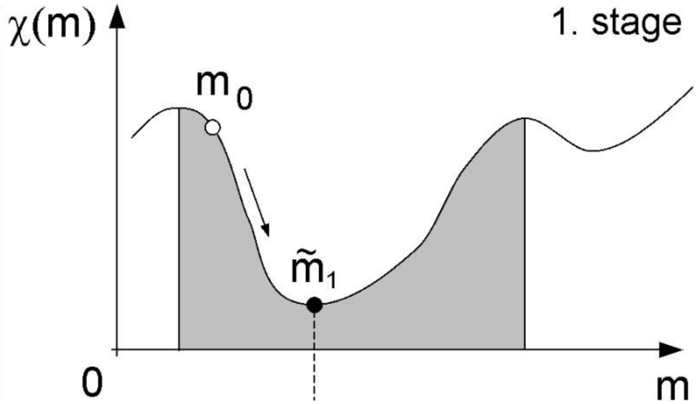


Bleibinhaus et al., 2007

Misfit minimisation: Multi-scale approach

- Sufficiently good initial models are often not available.
- The multi-scale approach is an empirical strategy that helps to overcome this problem

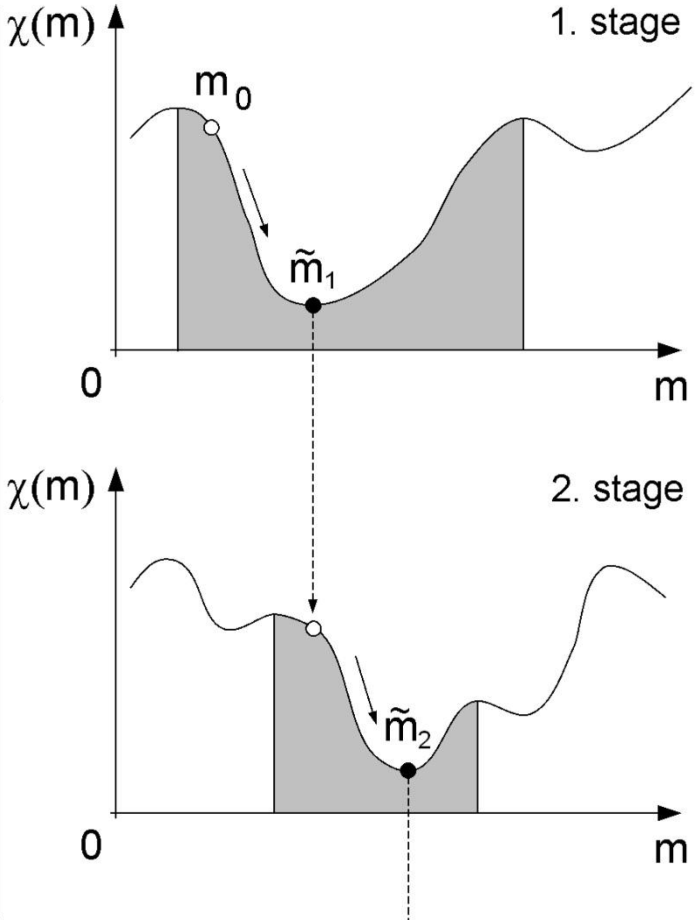
Misfit minimisation: Multi-scale approach



long-period data

→ long-wavelength structure

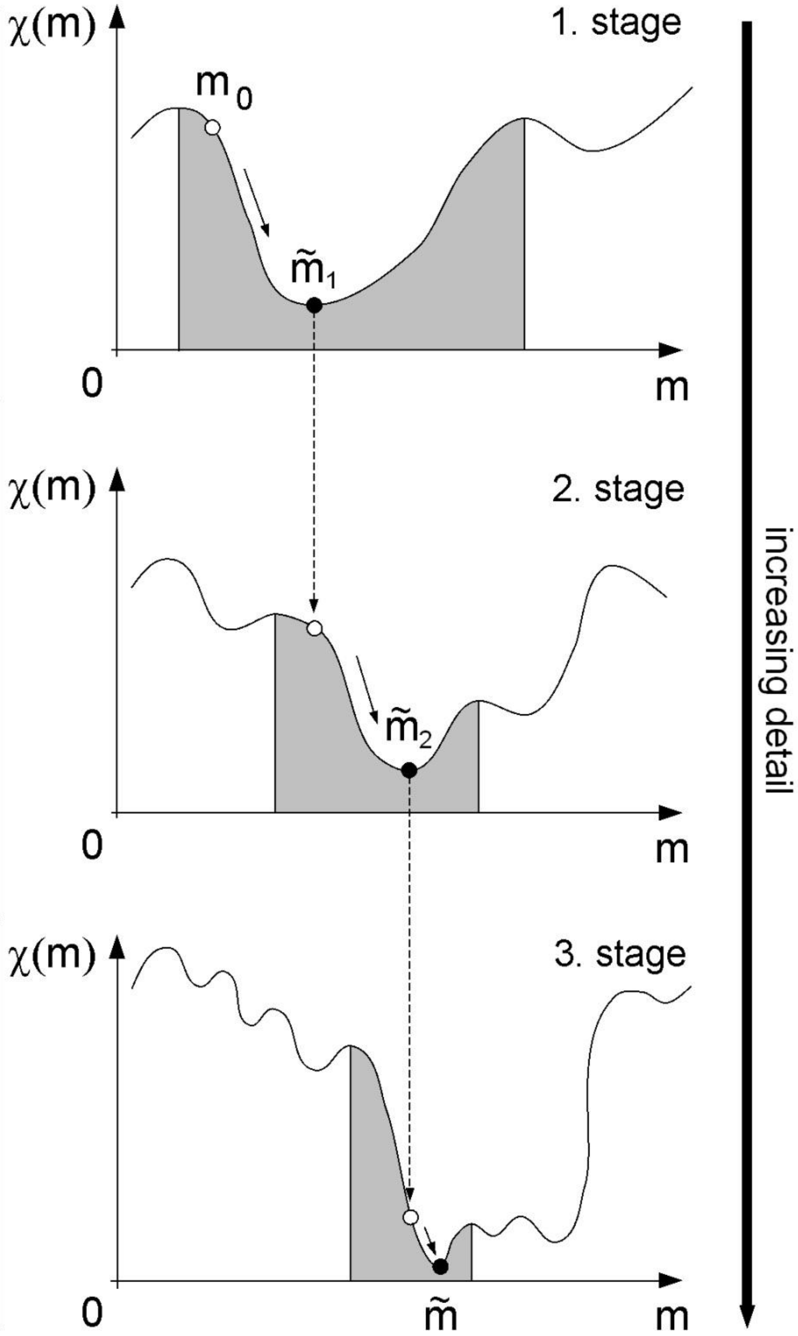
Misfit minimisation: Multi-scale approach



shorter-period data

→ shorter-wavelength structure

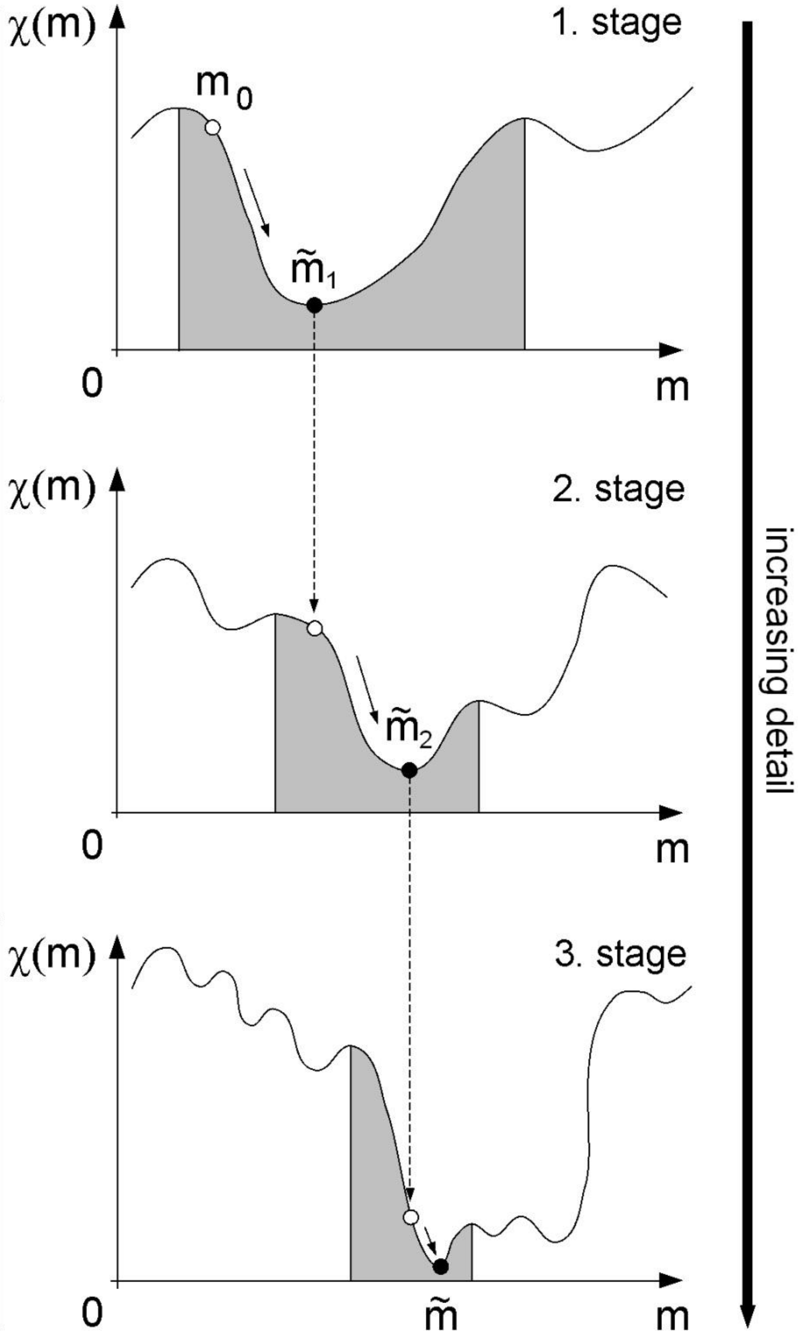
Misfit minimisation: Multi-scale approach

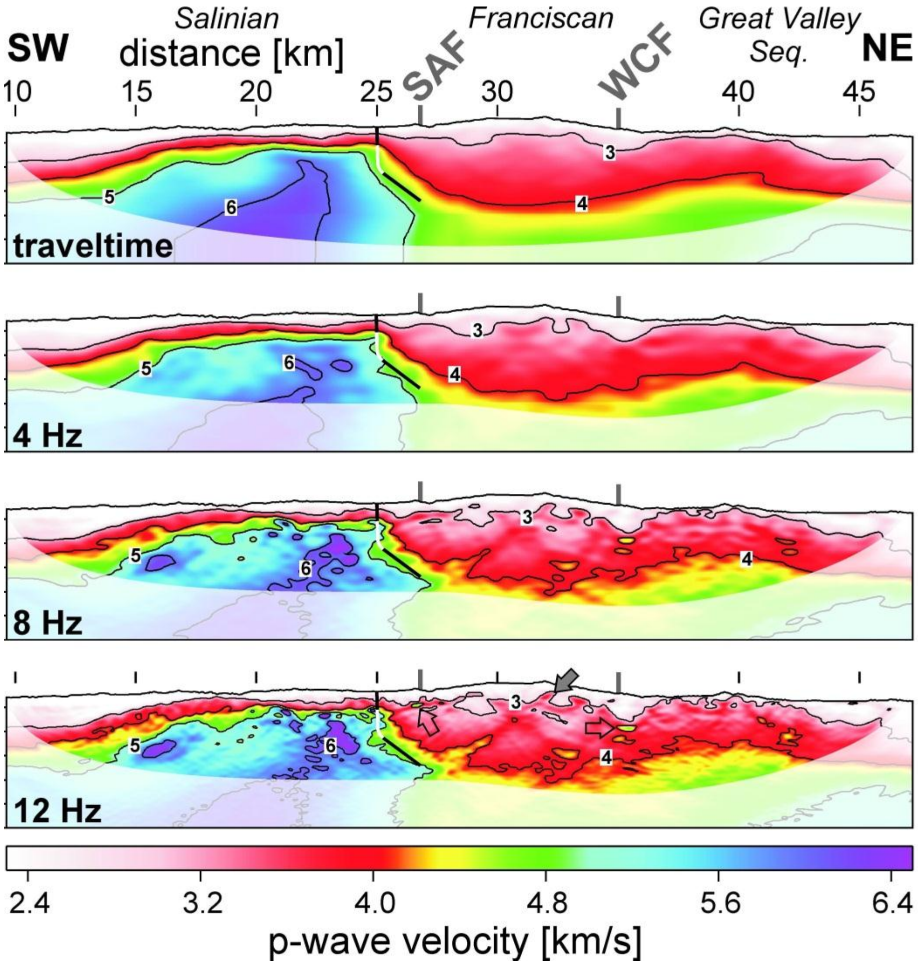


short-period data

→ short-wavelength structure

Misfit minimisation: Multi-scale approach





Bleibinhaus et al., 2007

4. Efficient computation of the gradient: The adjoint method

The adjoint method: Problem statement

Gradient-based methods

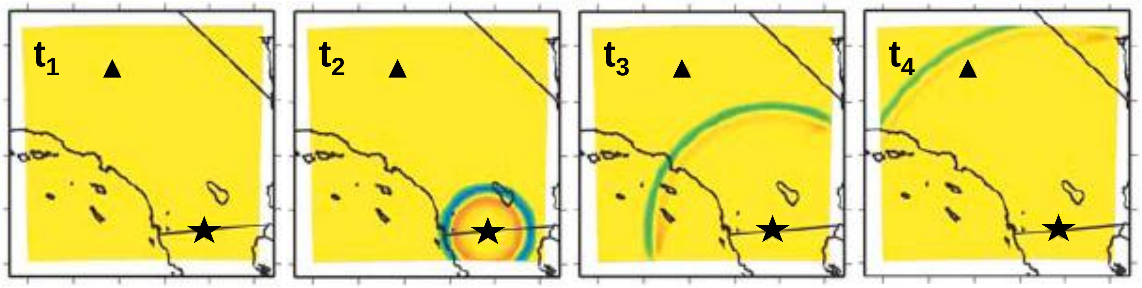
rely on the gradient of the misfit functional χ(m)

with respect to the model parameters m_i :

$$\boxed{h_i \propto -\frac{\partial \chi}{\partial m}}$$

How can we compute this quantity most efficiently?

1. Solve the forward problem



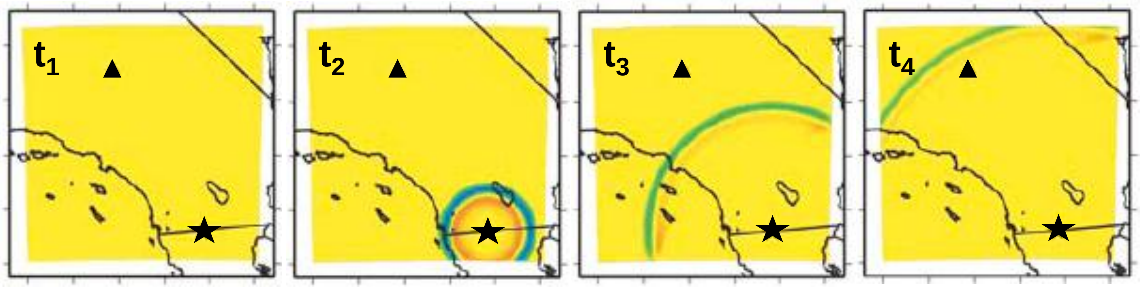
➡

forward field u
synthetic seismograms

Tape et al., 2007

The adjoint method: Adjoint method recipe

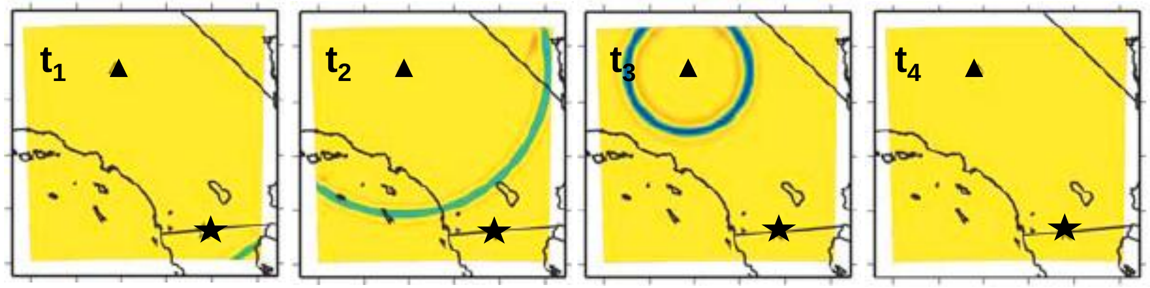
1. Solve the forward problem



forward field u
synthetic seismograms

2. Evaluate the misfit χ

3. Solve the adjoint problem
   - also a wave equation
   - runs backwards in time away from the receiver
   - source determined by the misfit

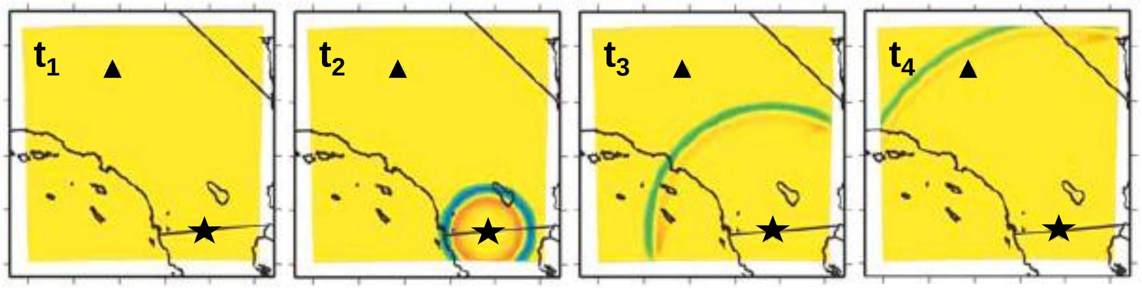


adjoint field u^t

Tape et al., 2007

The adjoint method: **Adjoint method recipe**

1. Solve the forward problem



forward field u
synthetic seismograms

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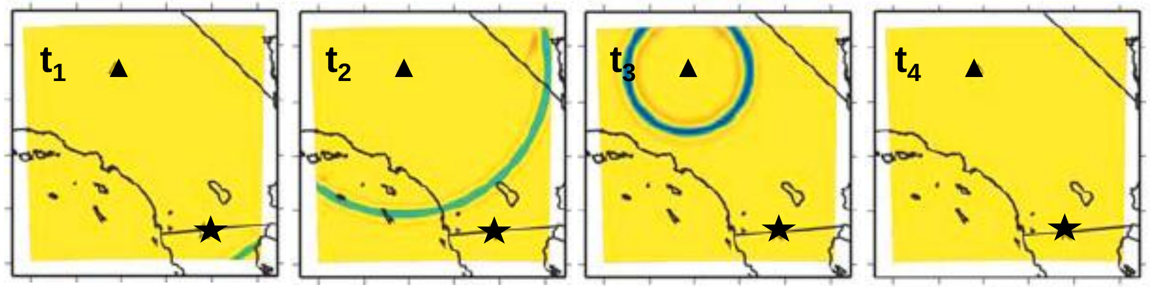


adjoint field u^t

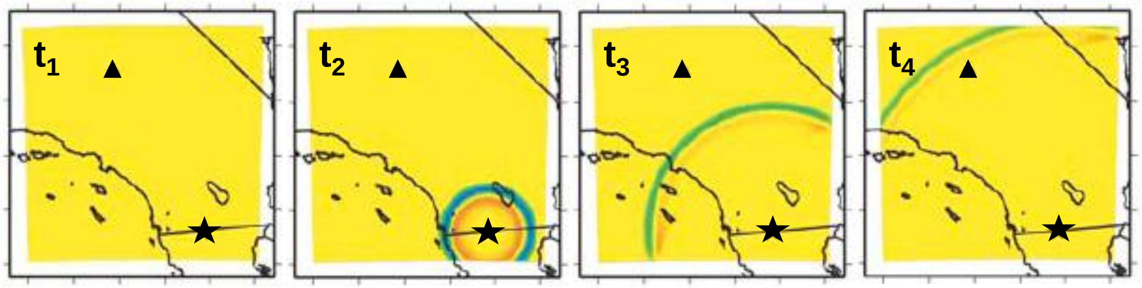
4. Compute the gradient by correlating u and u^t

$$\frac{\partial \chi(\mathbf{m})}{\partial \mathbf{m}_i} = \int_{\mathbf{Earth}} \int_{\mathbf{time}} [\mathbf{u} * \mathbf{u}^t]\, \mathbf{dt}\, \mathbf{d}^3\mathbf{x}$$

Tape et al., 2007

The adjoint method: Adjoint method recipe

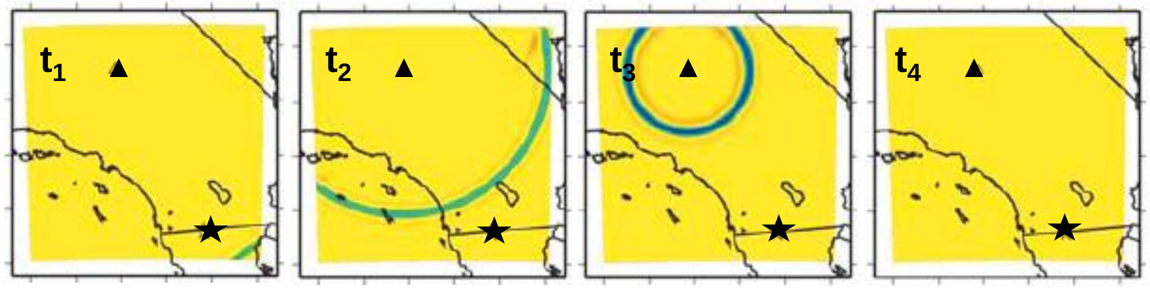
1. Solve the forward problem



forward field u
synthetic seismograms

2. Evaluate the misfit χ

3. Solve the adjoint problem
   - also a wave equation
   - runs backwards in time away from the receiver
   - source determined by the misfit



adjoint field u^t

4. Compute the gradient by correlating u and u^t

Fréchet kernel

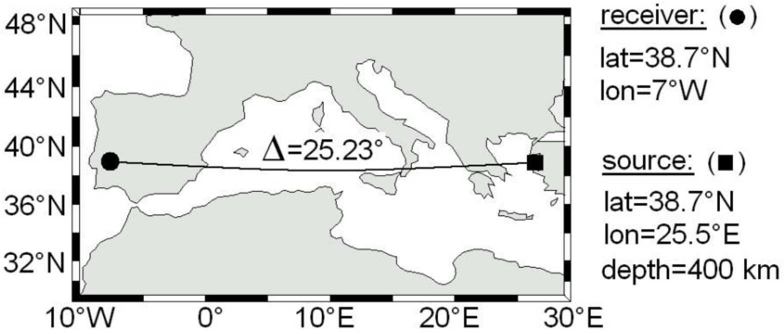
sensitivity kernel

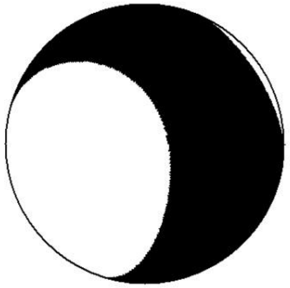
sensitivity density

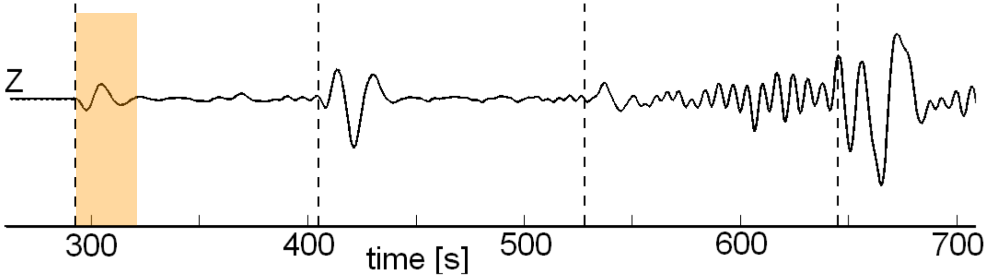
$$\frac{\partial\chi(\mathbf{m})}{\partial\mathbf{m}_i} = \int_{\mathbf{Earth}} \int_{\mathbf{time}} [\mathbf{u} * \mathbf{u}^t]\, \mathbf{dt}\, \mathbf{d}^3\mathbf{x}$$

Tape et al., 2007

The adjoint method: **Fréchet kernel gallery**

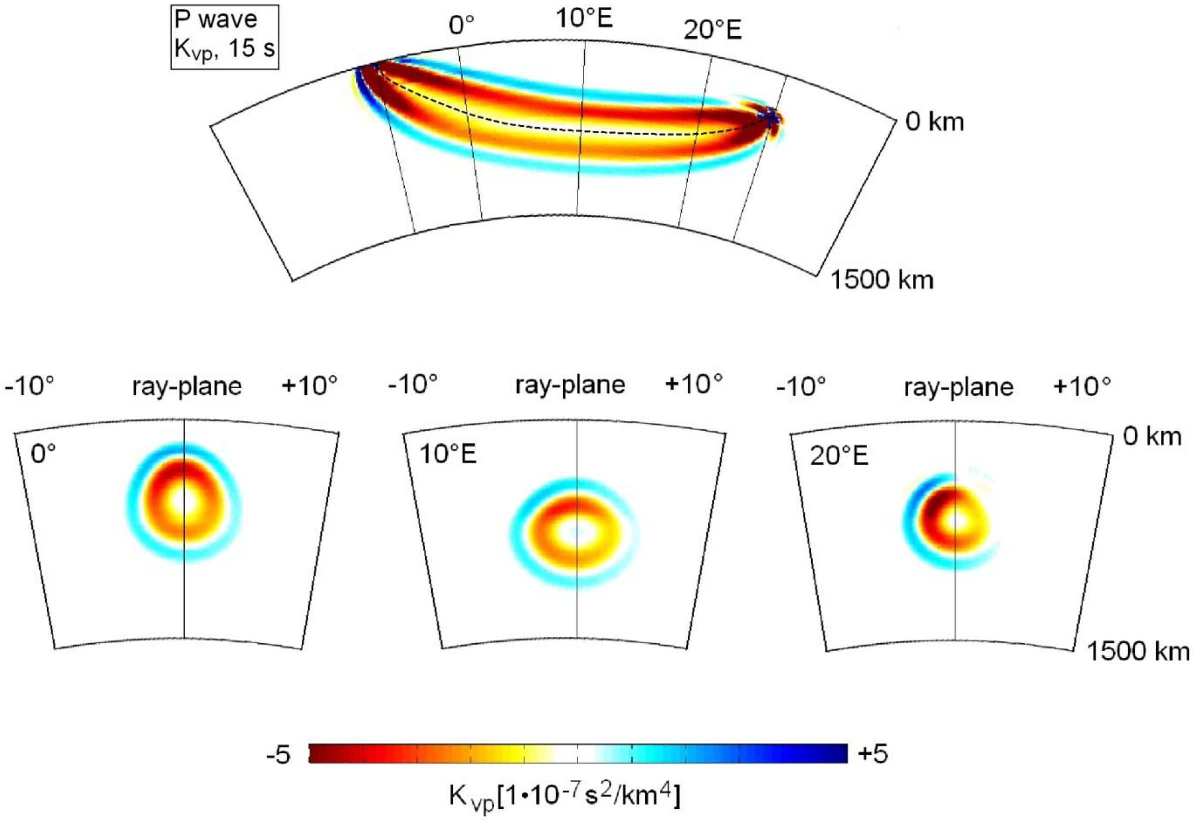




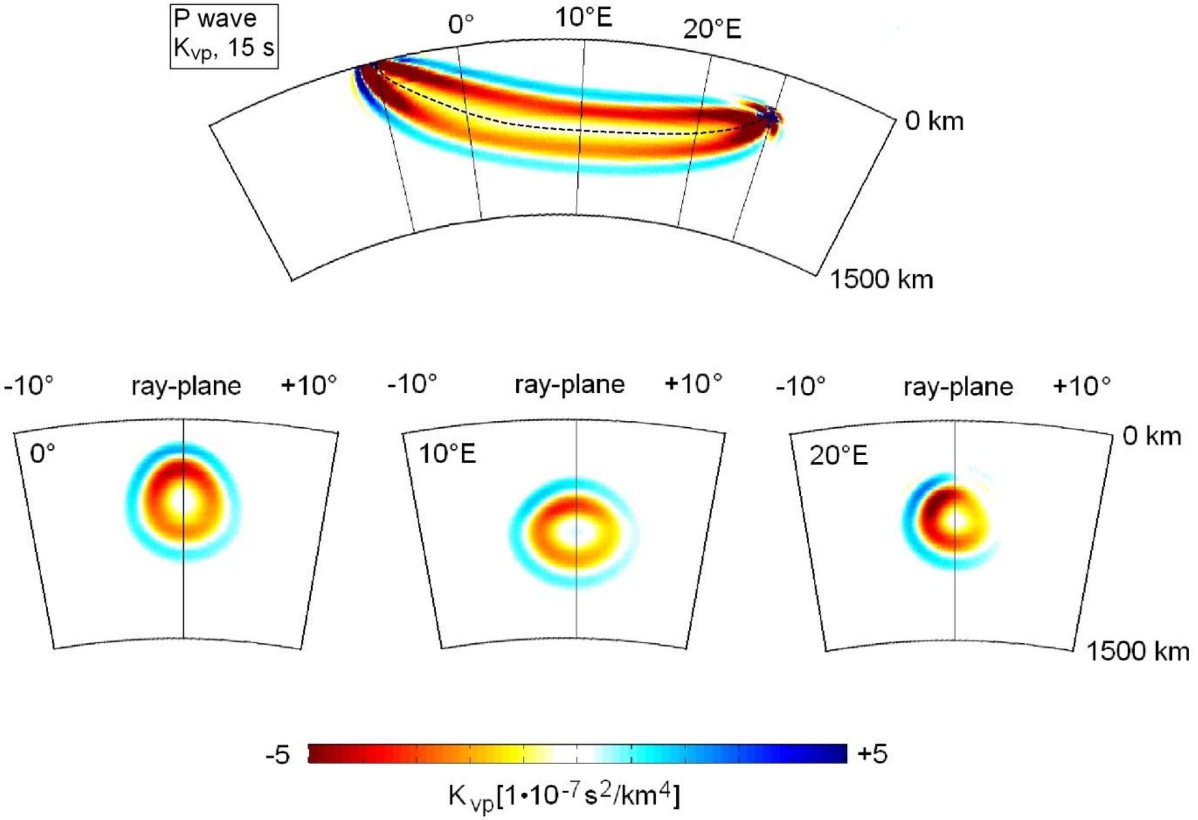


measurement: cross-correlation time shift

The adjoint method: Fréchet kernel gallery



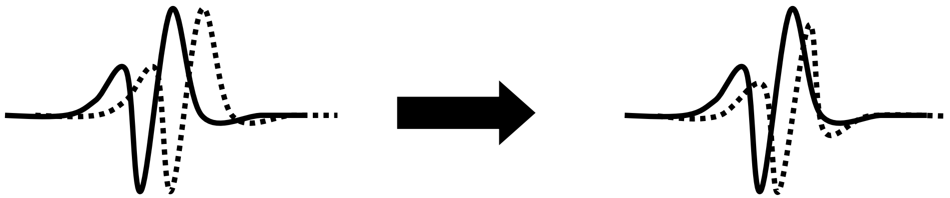
The adjoint method: Fréchet kernel gallery



—— data

········ synthetic

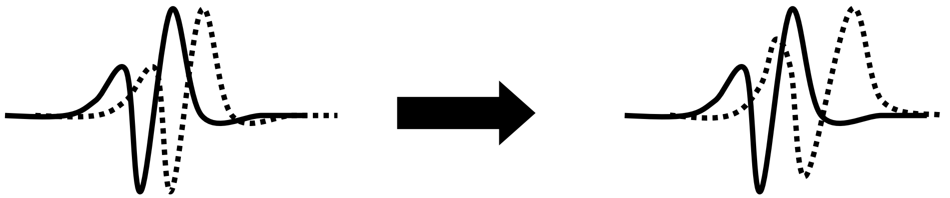
red: $\Delta v_p>0$



synthetics appear earlier

time shift smaller

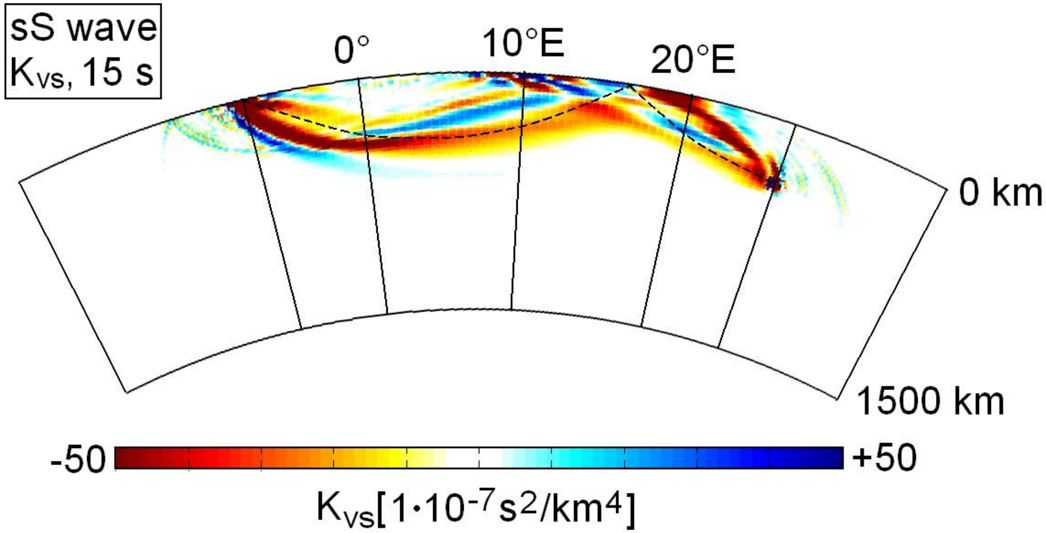
blue: $\Delta v_p>0$

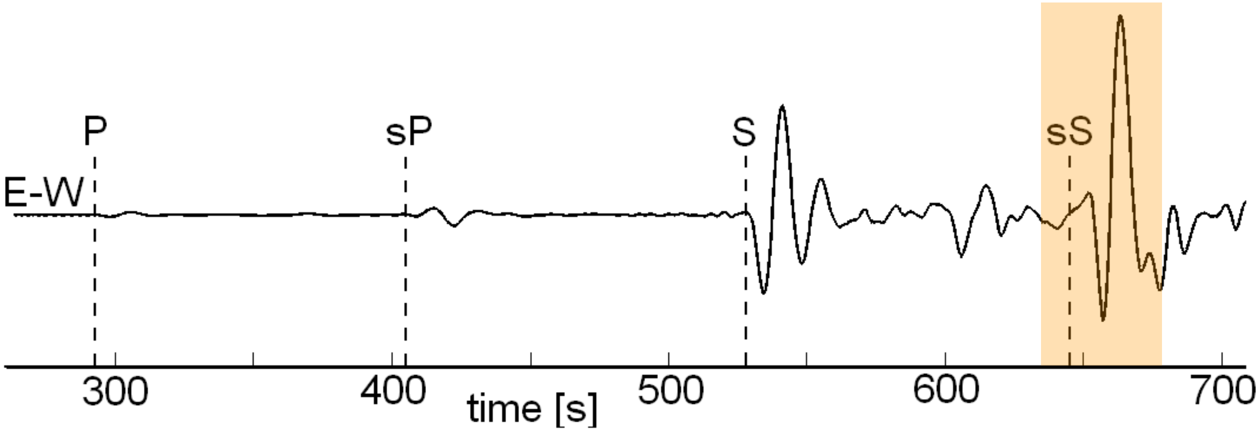


synthetics appear later

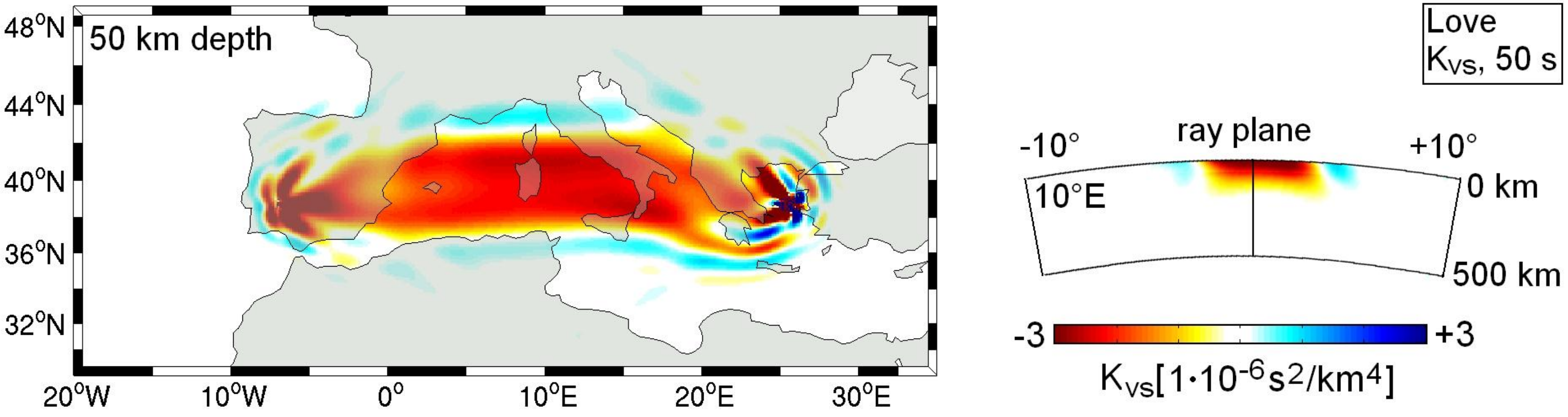
time shift larger

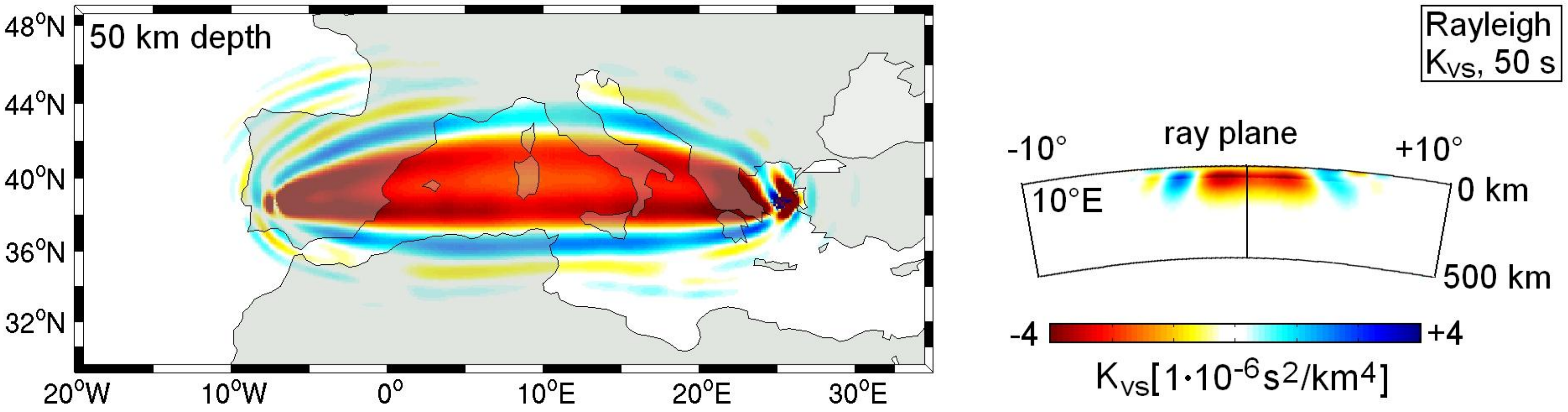
The adjoint method: Fréchet kernel gallery





The adjoint method: **Fréchet kernel gallery**

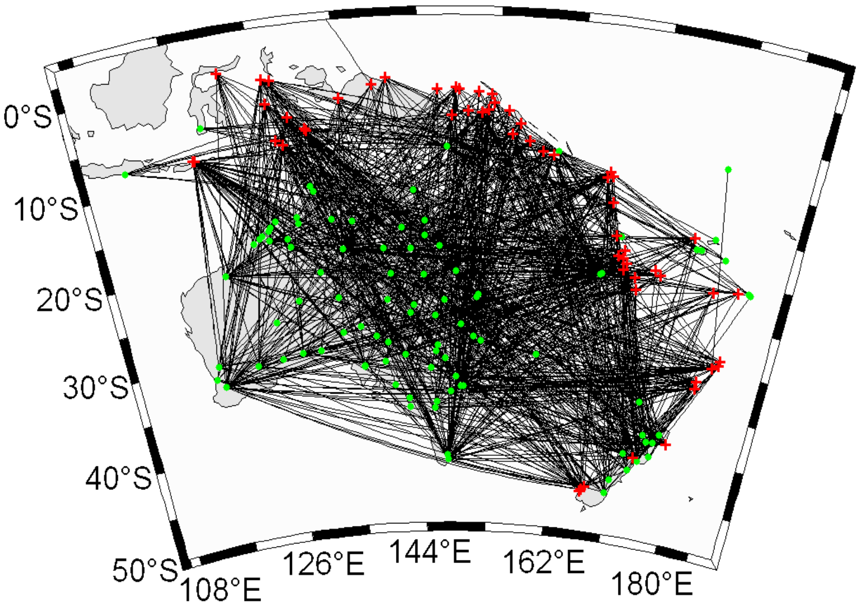




5. Applications

Applications: Continental scale

ray coverage



- 60 earthquakes in the Australasian region
- data

  fundamental- and higher-mode surface waves

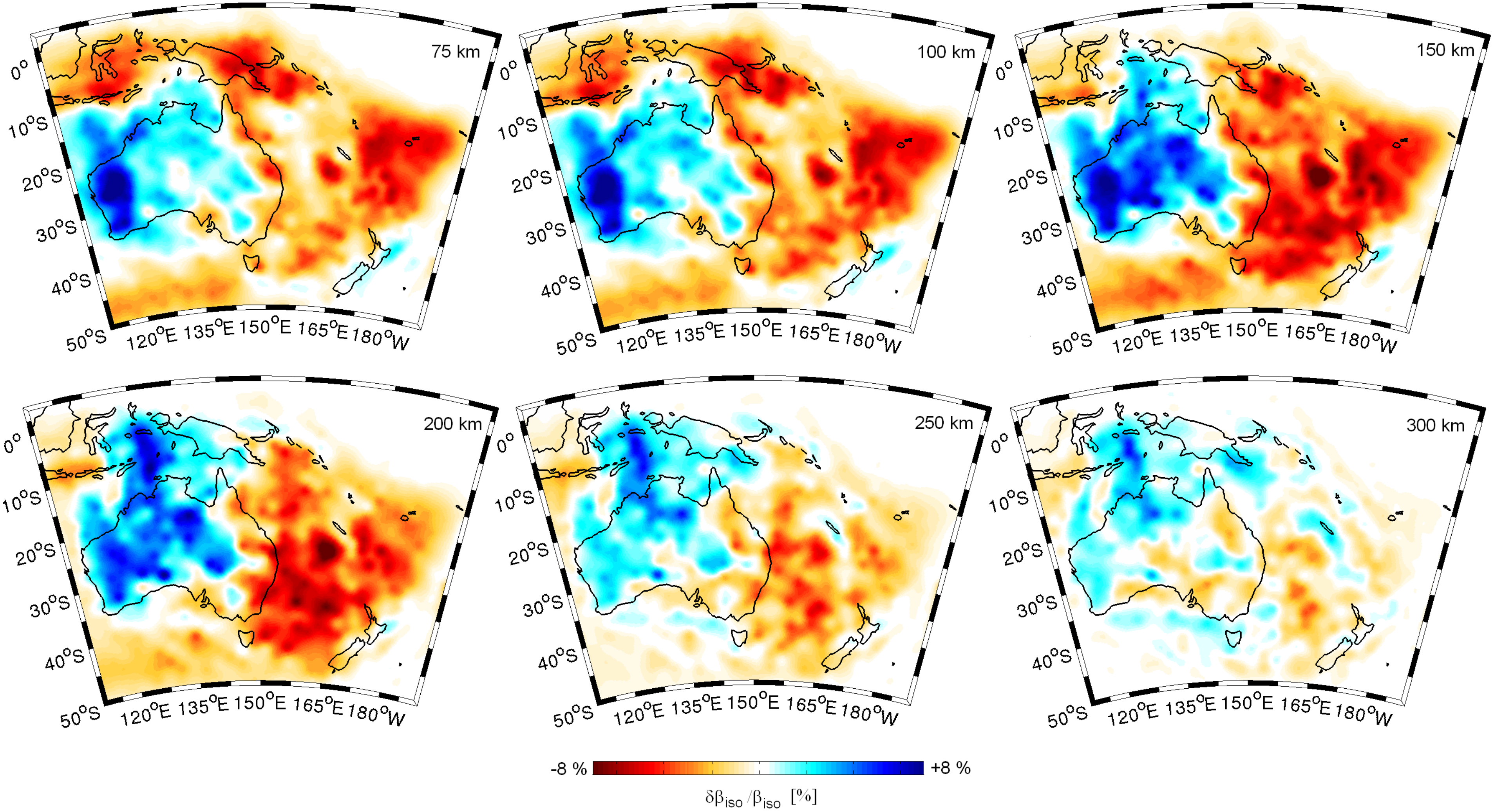
  long-period body waves

  unidentified phases
- periods between 30 s and 200 s

- spectral-element simulations
- measurements of time-frequency phase misfit
- 19 conjugate-gradient iterations

Applications: Continental scale

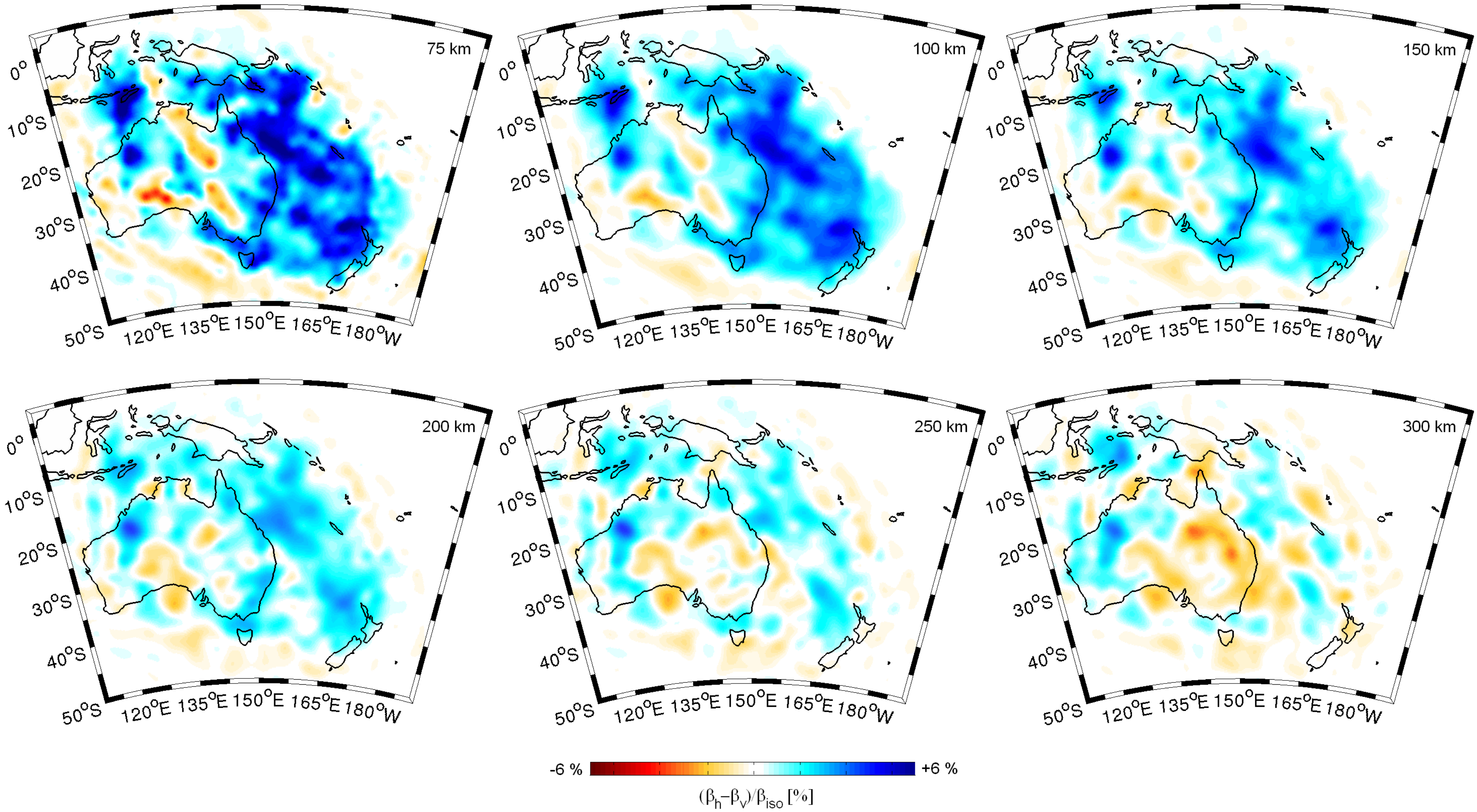
isotropic S wave speed



Fichtner, Kennett, Igel, Bunge: EPSL 2010

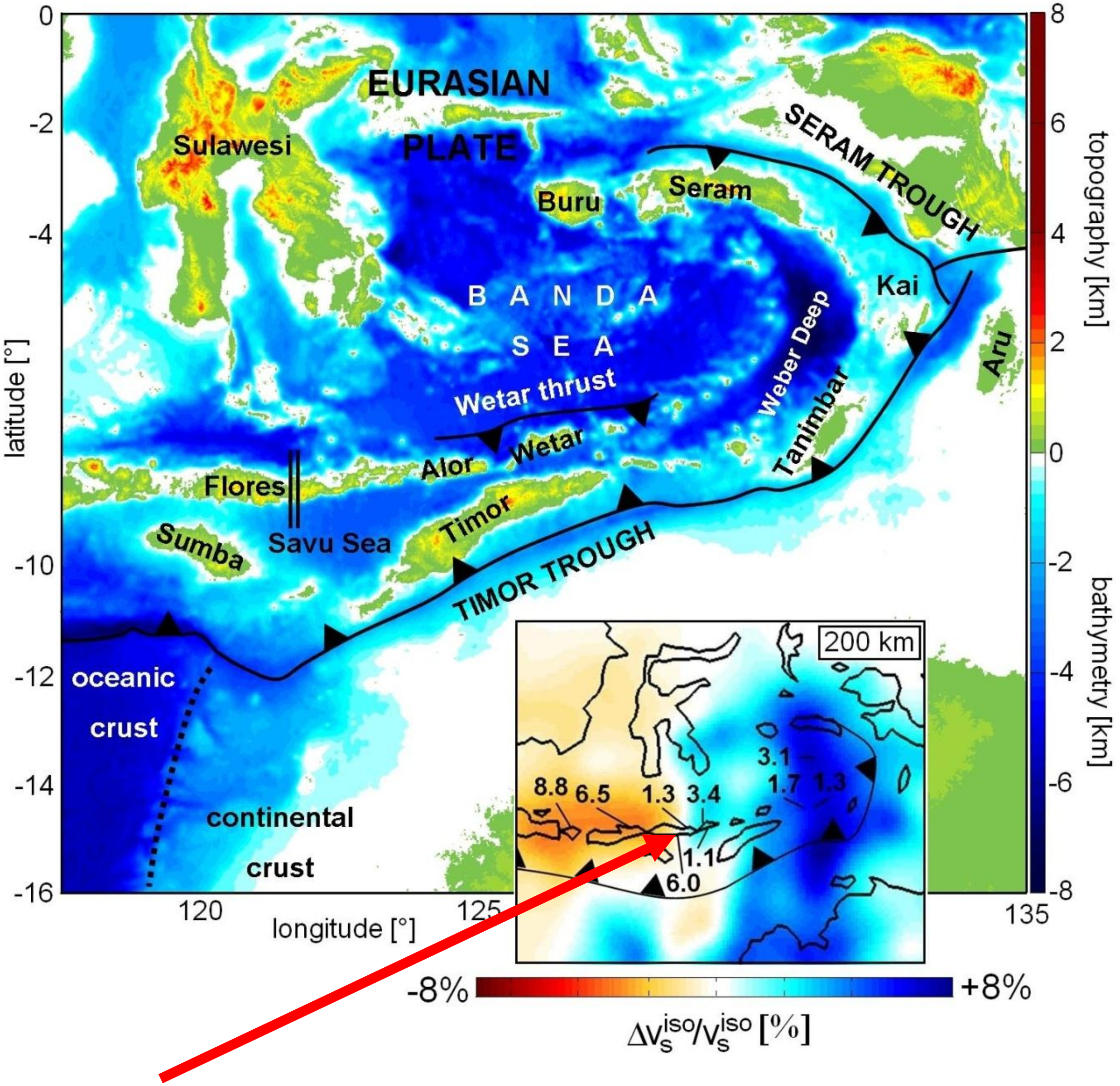
Applications: Continental scale

radial anisotropy $(V_{sh}-V_{sv})/V_s^{iso}$



Fichtner, Kennett, Igel, Bunge: EPSL 2010

Applications: Continental scale



- **Superimposed:**

^{3}He/^{4}He ratios from arc volcanics

[Hilton & Craig, 1989, Hilton et al., 1992]

- **Observation at 200 km depth:**

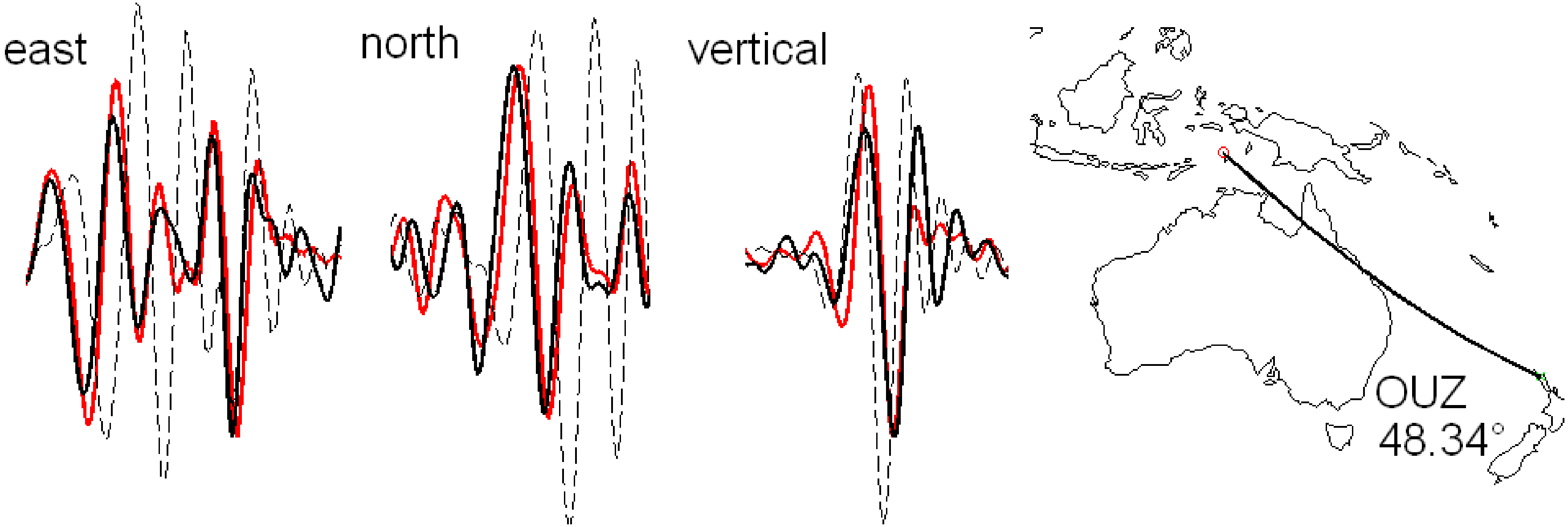
low to **high velocities**

high to **low He ratios**

Old continental lithosphere is subduced to more than 200 km depth !!!

Fichtner, de Wit, van Bergen: EPSL 2010

Applications: Continental scale



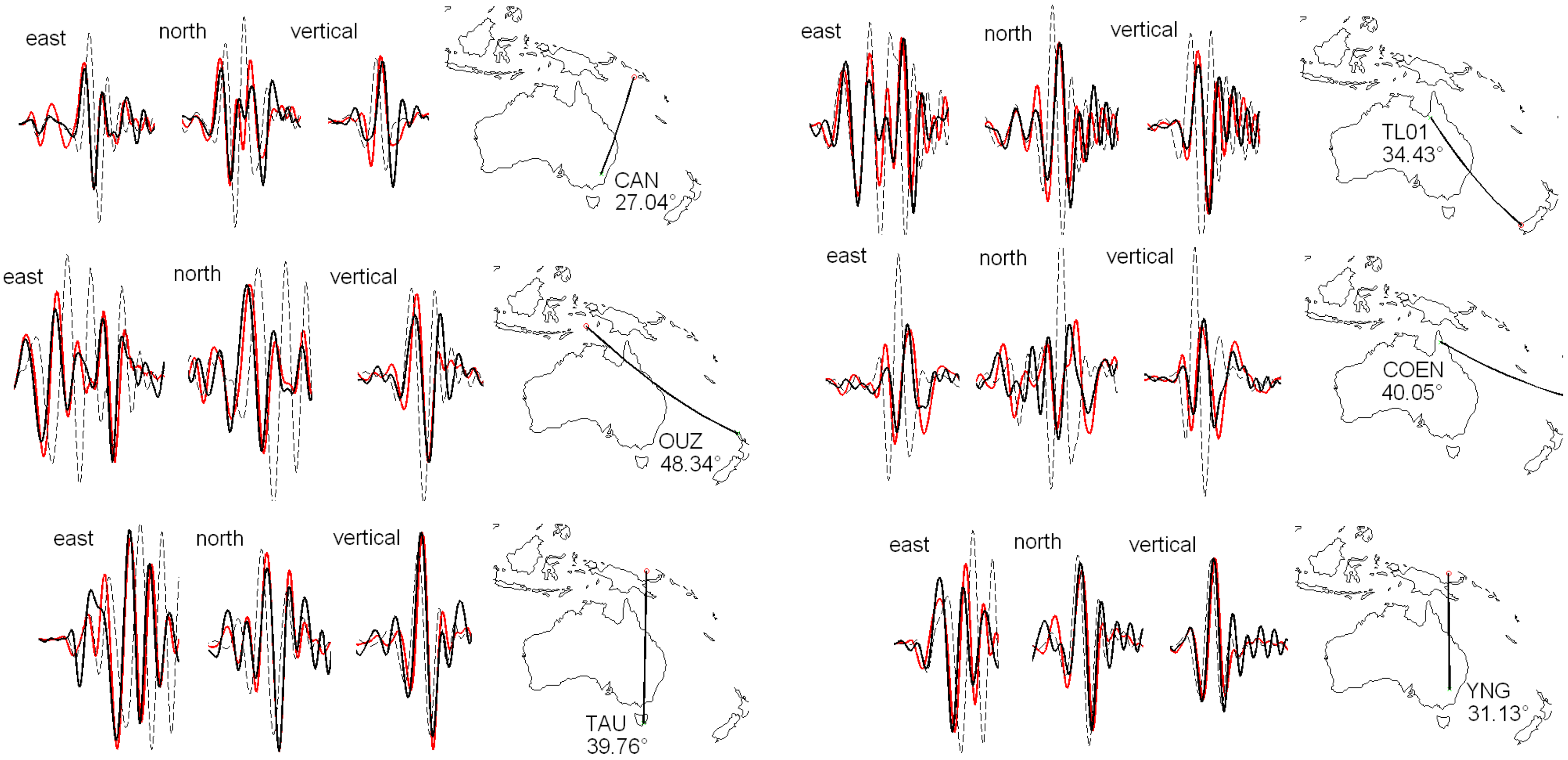
data

initial model

final model

T = 30 s

Applications: Continental scale



- **accurate determination of seismic source characteristics** *
- **long-term goal: improved tsunami warning**

* Hingee, Tkalcic, Fichtner, Sambride: GJI 2010

6. Challenges and directions of future research

Challenges and directions of future research

1. Quantification of resolution

Challenges and directions of future research

1. Quantification of resolution

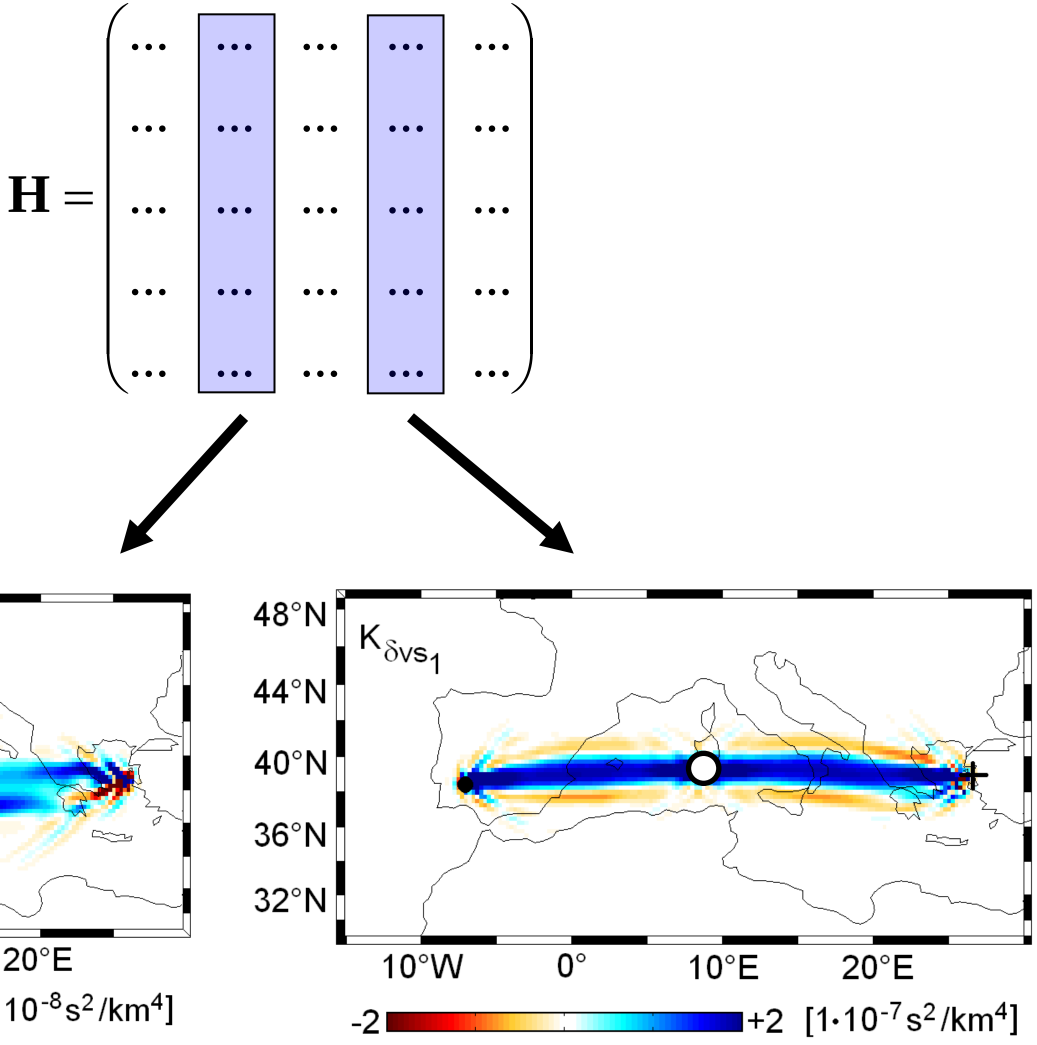
- **Physics:**

second-order scattering

- **Inversion:**

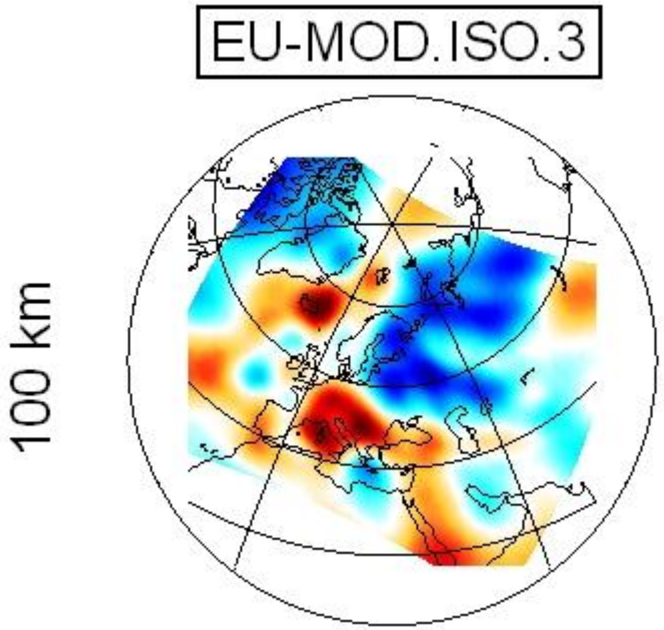
covariance information

Hessian of the misfit functional:

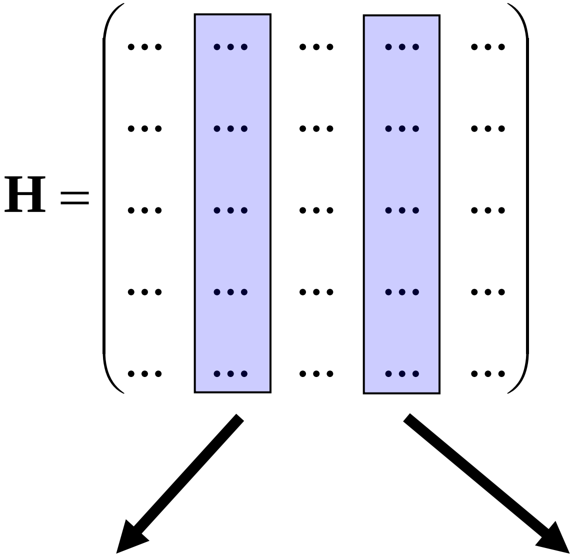


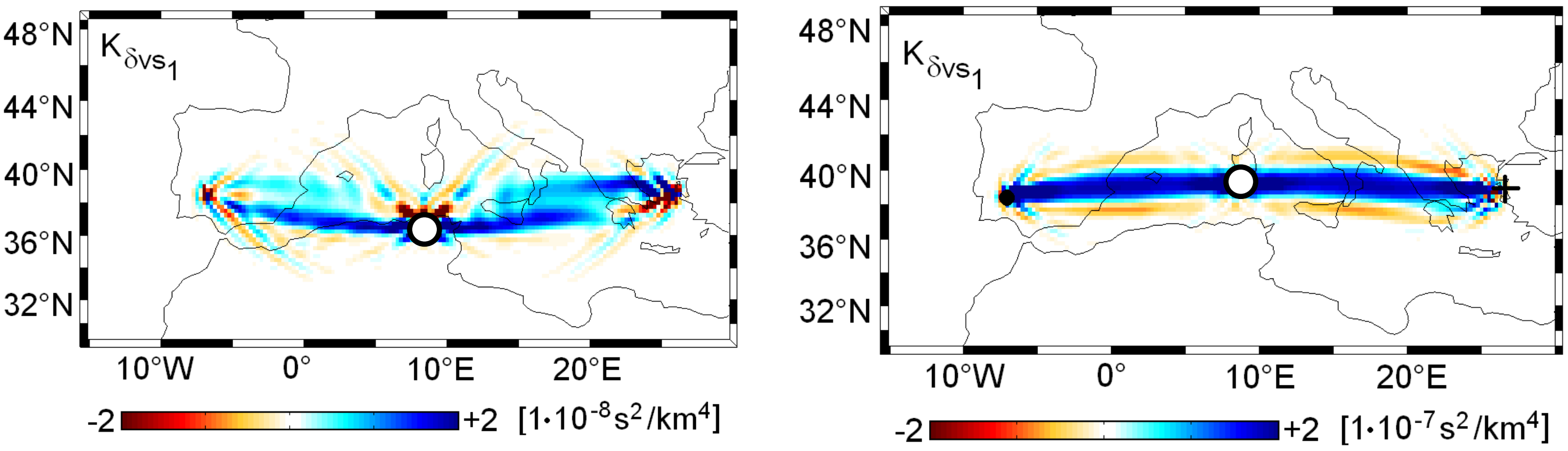
Challenges and directions of future research

1. Quantification of resolution



Hessian of the misfit functional:





Challenges and directions of future research

1. Quantification of resolution

2. Multi-parameter inversions
    - Q
    - anistropy
    - density

3. More efficient optimisation schemes

4. Design of misfit functionals that extract target-oriented information

5. Combination of full waveform inversion with noise tomography

6. …

Much of this can soon be found in:

Numerical solution of the elastic wave equation

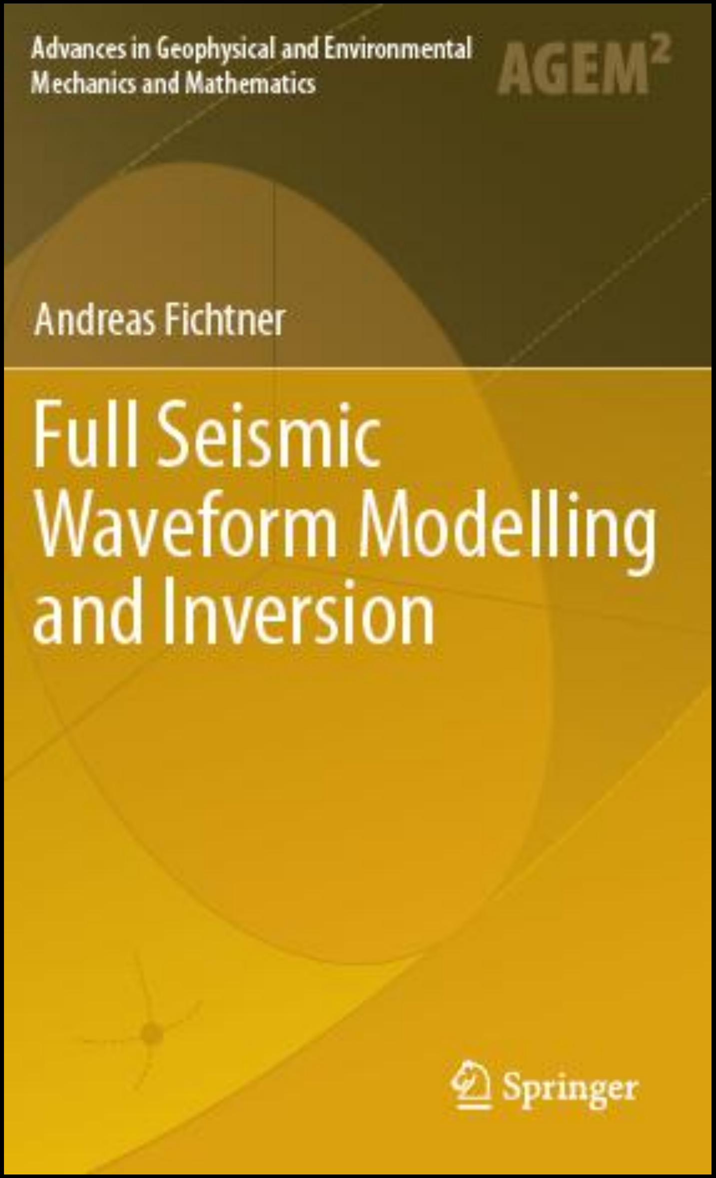
- Finite-difference methods
- Spectral-element methods
- Absorbing boundaries
- Visco-elastic dissipation

Iterative solution of the inverse problem

- Introduction to iterative nonlinear minimisation
- The continuous adjoint method
- First and second derivatives
- The discrete adjoint method
- Misfit functionals and adjoint sources
- Fréchet and Hessian kernel gallery

Applications

- Full waveform tomography on continental scales
- Application of full waveform tomography to active-source surface-seismic data [by F. Bleibinhaus]
- Source stacking data reduction for full waveform tomography at the global scale [by Y. Capdeville]



Advances in Geophysical and Environmental
Mechanics and Mathematics

AGEM²

Andreas Fichtner

Full Seismic
Waveform Modelling
and Inversion

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Thank you for your attention!