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Comparison of Accuracy of the FDM, FEM, SEM and DGM

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the overall accuracy of a numerical scheme

for a given space-time grid depends mainly on accuracy

- in
  - a homogeneous medium
    - V_P/V_S ratio
  - a smoothly spatially varying medium
    - spatial variability of material parameters

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- at
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    geometry, continuity of displacement and traction
  - a free surface

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    - geometry, zero traction
- of
  - a grid boundary
    - transparency or symmetry
  - simulation of source
    - location, mechanism, time function
  - incorporation of attenuation
    - frequency dependence, spatial variability

here we focus only on the accuracy
in the homogeneous medium
and, specifically,

on the accuracy
with respect to V_P / V_S ratio

why ?

because in surface sediments

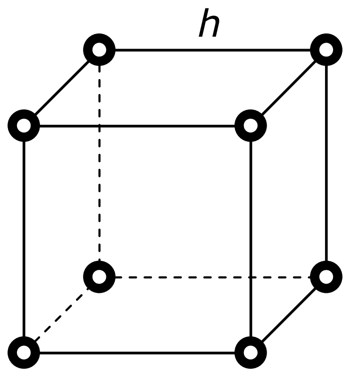
and, mainly,

in sedimentary basins and valleys

often $V_P / V_S > 5$

spatial grids

conventional

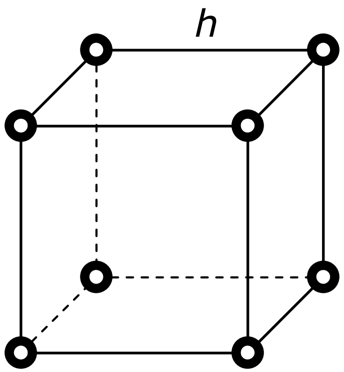


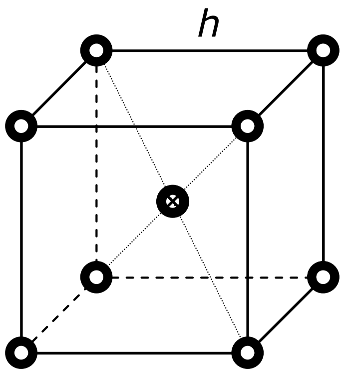
u_x, u_y, u_z

spatial grids

conventional

partly staggered



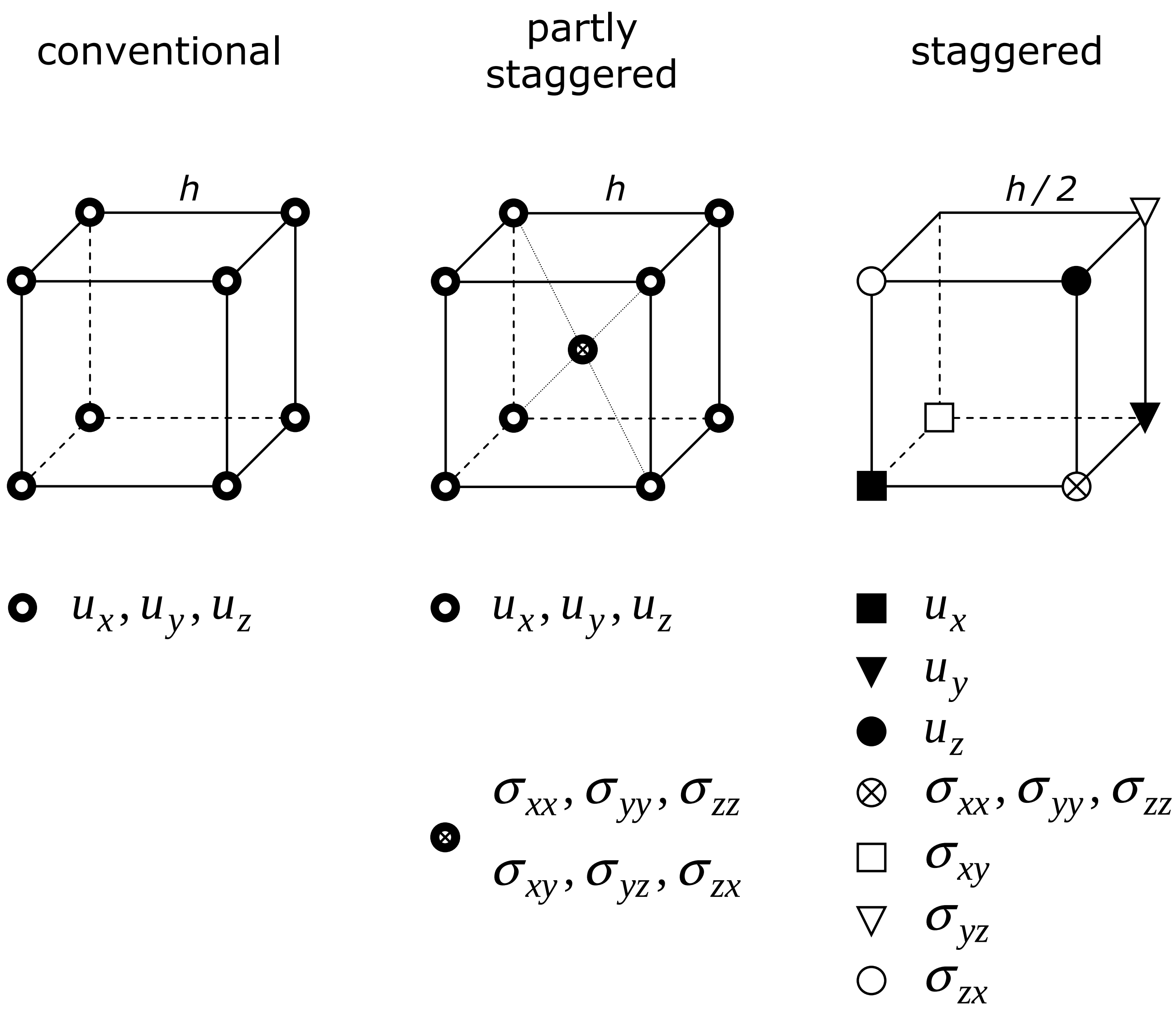


○ u_x, u_y, u_z

○ u_x, u_y, u_z

⊗ $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$
$\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$

spatial grids



3D numerical schemes					
method		**equation formulation**	**grid**	**add. specif.**	**order**
FD D CG 2	finite-difference	displacement	conventional		
FD DS PSG 2		displacement -stress	partly staggered		
FD DS SG 2		displacement -stress	staggered		

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FE L8	finite-element	displacement	conventional	Lobatto 8-point integr.	
FE G1				Gauss 1-point integr.	
FE G8				Gauss 8-point integr.	

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FD D CG 4a	finite-difference	displacement	conventional		4
FD D CG 4b					
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<table>
<tr><th colspan="6">3D numerical schemes</th></tr>
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<tr><td>FD D CG 4b</td></tr>
<tr><td>FD DS SG 4</td><td>displacement-stress</td><td>staggered</td></tr>
<tr><td>SE 4 cn, vn</td><td>spectral-element</td><td>displacement</td><td>conventional</td><td>GLL integr.</td></tr>
</table>

assuming

an unbounded homogeneous isotropic elastic medium

and

a uniform cubic grid

we wrote all schemes in a unified form:

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we wrote all schemes in a unified form:

$$U\left(x, y, z; t+\Delta t \right) = \text{numerical\_scheme}\left\{ U\left(t-\Delta t \right), U\left(t \right) \right\}$$

FD D CG 2 = FE L8

FD D CG 2 = FE L8

FD DS PSG 2 = FE G1

FD D CG 2	**=**	**FE L8**
FD DS PSG 2	**=**	**FE G1**
DG CF 2	**=**	**FE G8**

numerical solution in one time step

$$U(x, y, z; t + \Delta t) = \text{numerical\_scheme}\left\{ U(t - \Delta t), U(t) \right\}$$

exact values

a relative local error in amplitude in one time step

A_N = numerical amplitude at $t + \Delta t$

A_E = exact amplitude at $t + \Delta t$

$$\varepsilon = \left(\frac{\Delta t_{ref}}{\Delta t} \right)^2 \frac{A_N - A_E}{A_E}$$

$\Delta t_{ref} = \Delta t$ for FD DS SG 4

$p = 0.9 \quad s = 1/6 \quad V_P / V_S = 1.42$

let us compare the schemes

with

the **usual** spatial discretizations

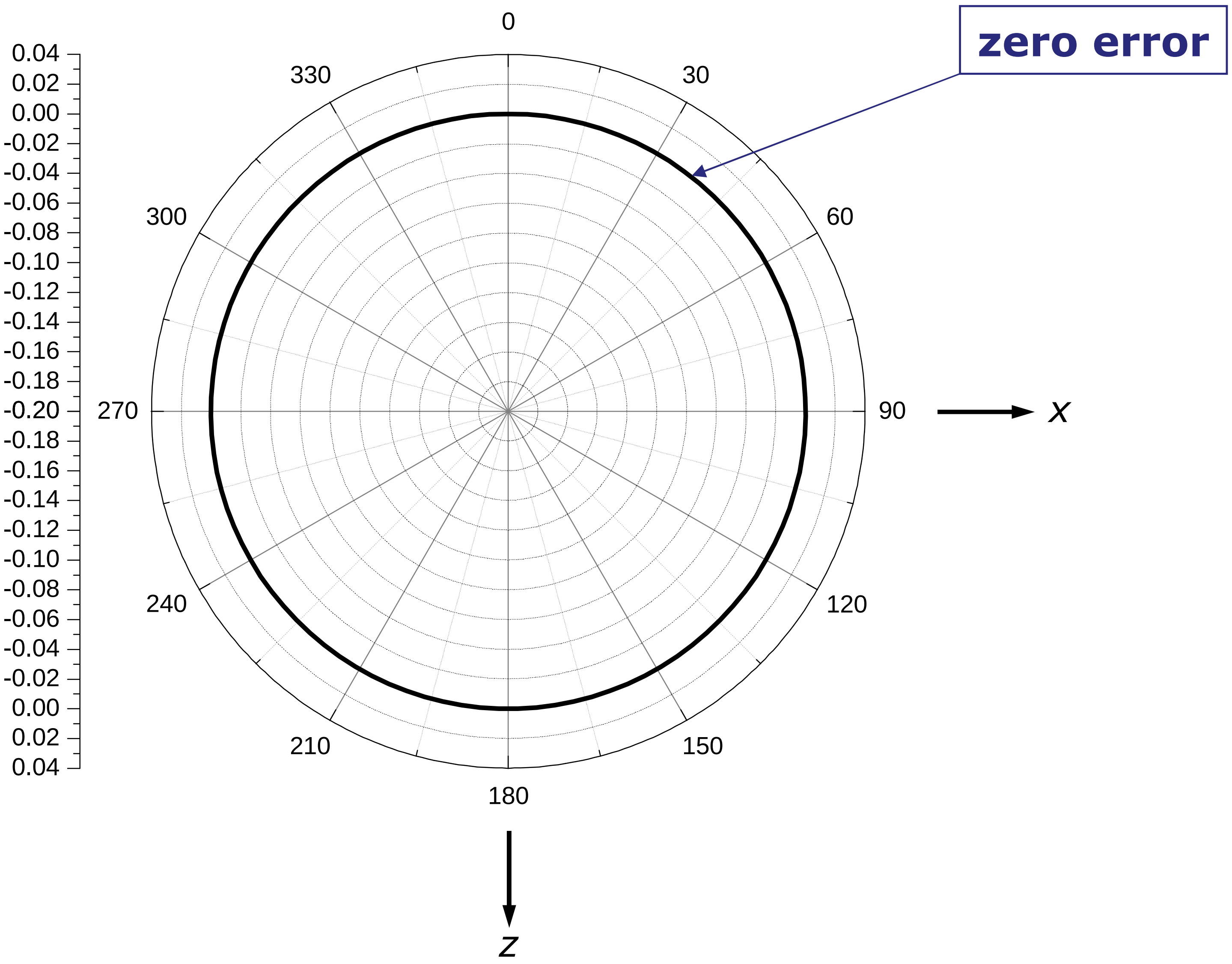
–

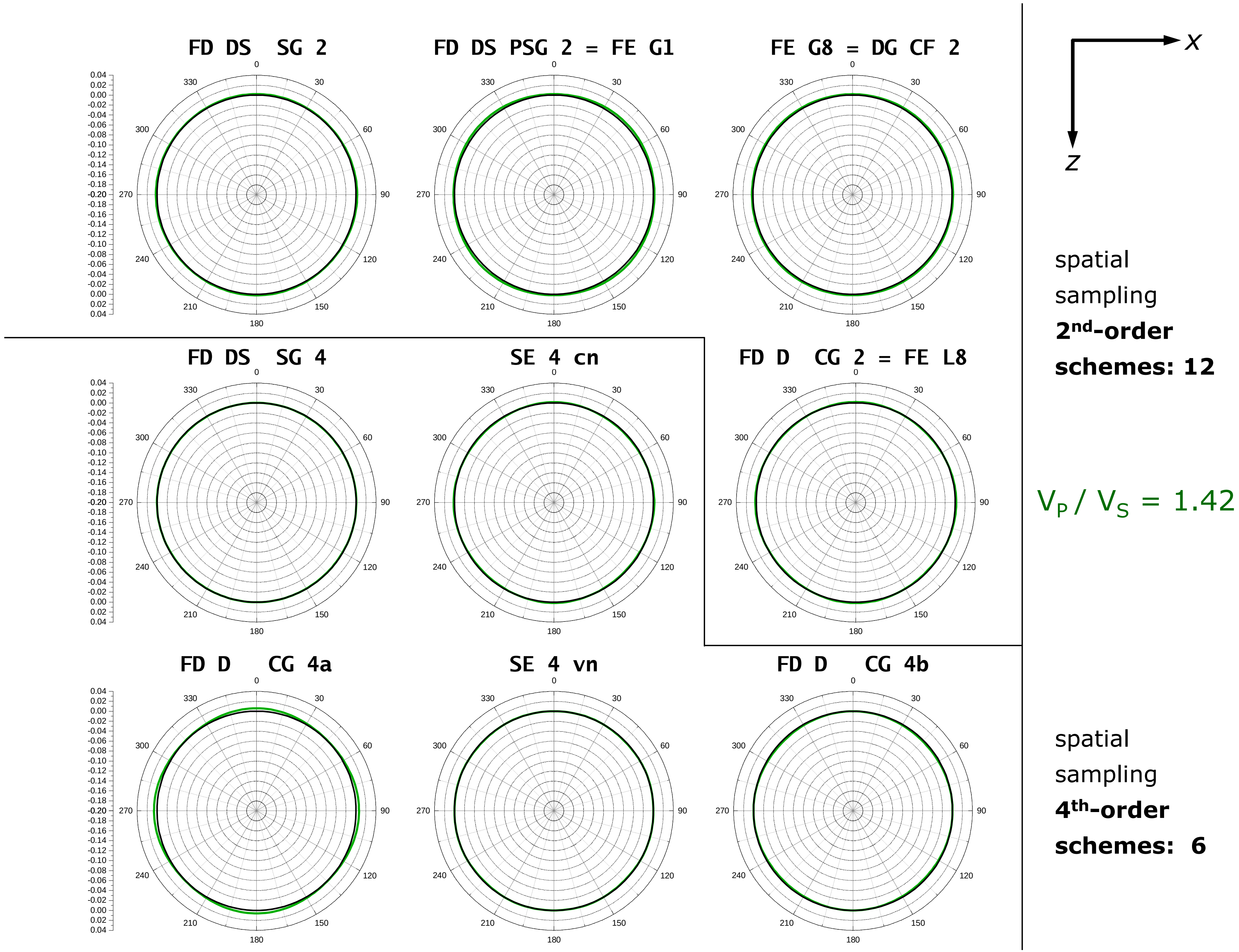
6 grid spacings per wavelength with the **4th-order** schemes

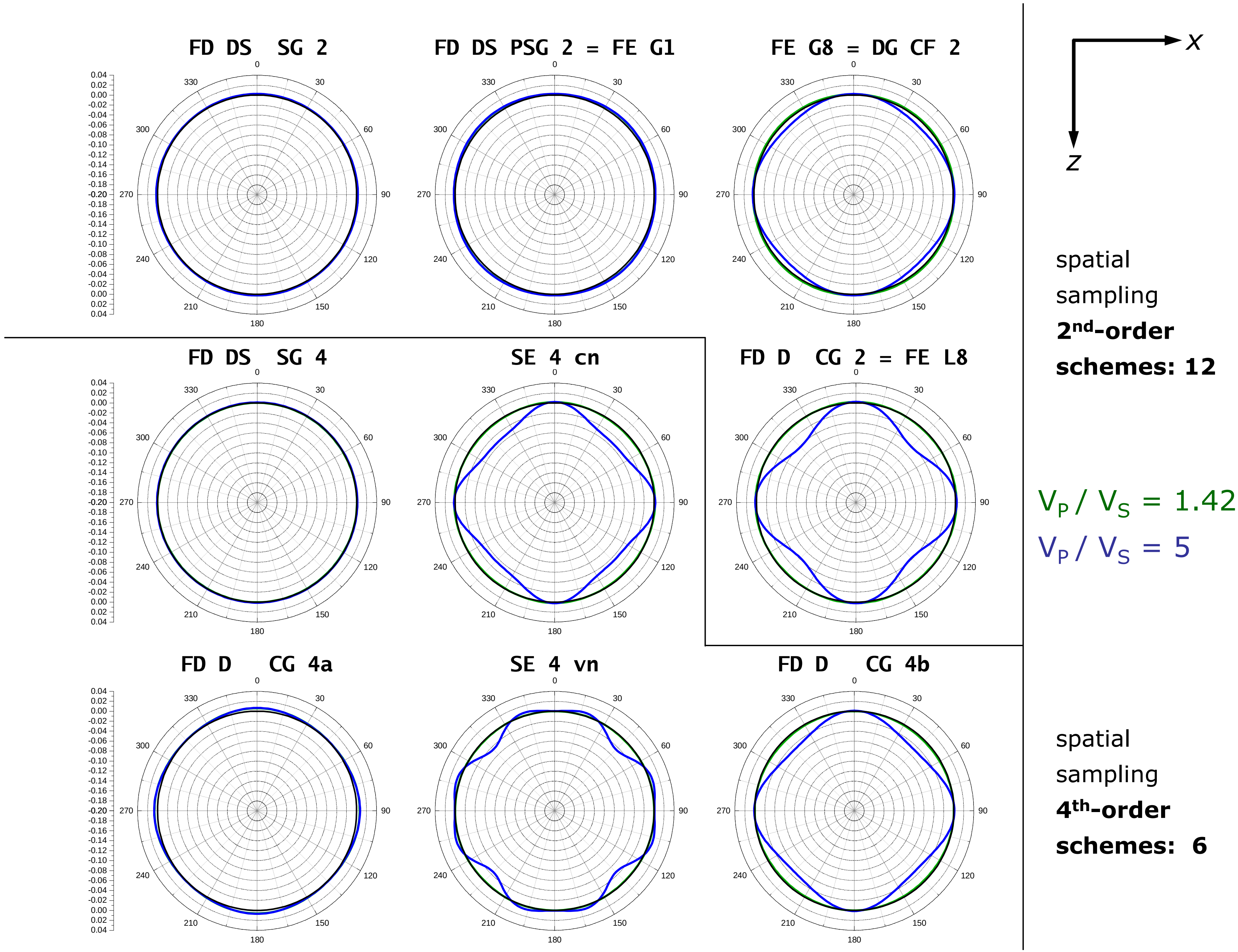
and

12 grid spacings per wavelength with the **2nd-order** schemes

local relative error in amplitude
for plane S waves propagating in
all directions of the xz-plane




FD DS SG 2
FD DS PSG 2 = FE G1
FE G8 = DG CF 2
FD DS SG 4
SE 4 cn
FD D CG 2 = FE L8
FD D CG 4a
SE 4 vn
FD D CG 4b
x
z
spatial sampling 2nd-order schemes: 12
VP / VS = 1.42
spatial sampling 4th-order schemes: 6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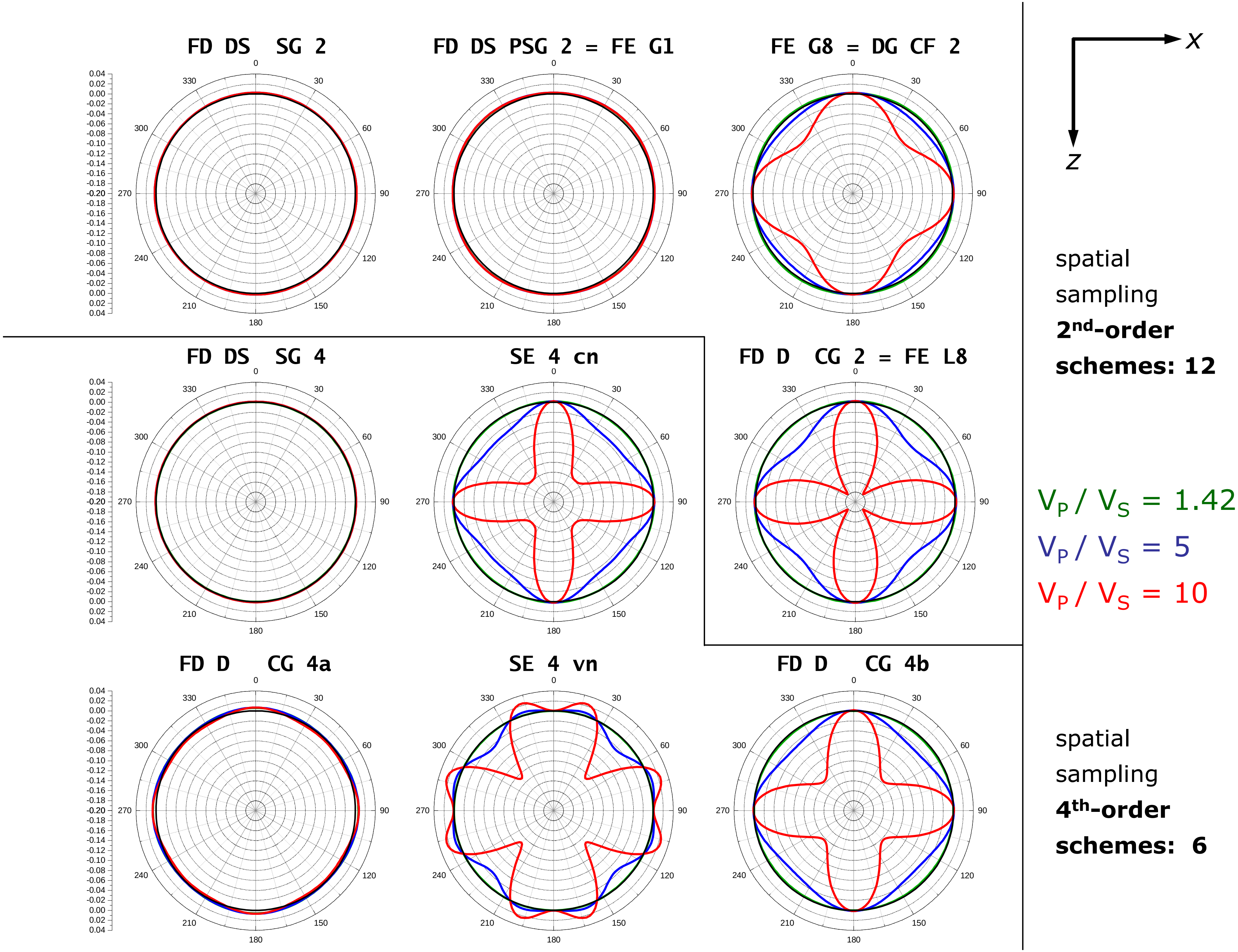
2nd-order schemes: 12
$V_P / V_S = 1.42$
$V_P / V_S = 5$
spatial sampling
4th-order schemes: 6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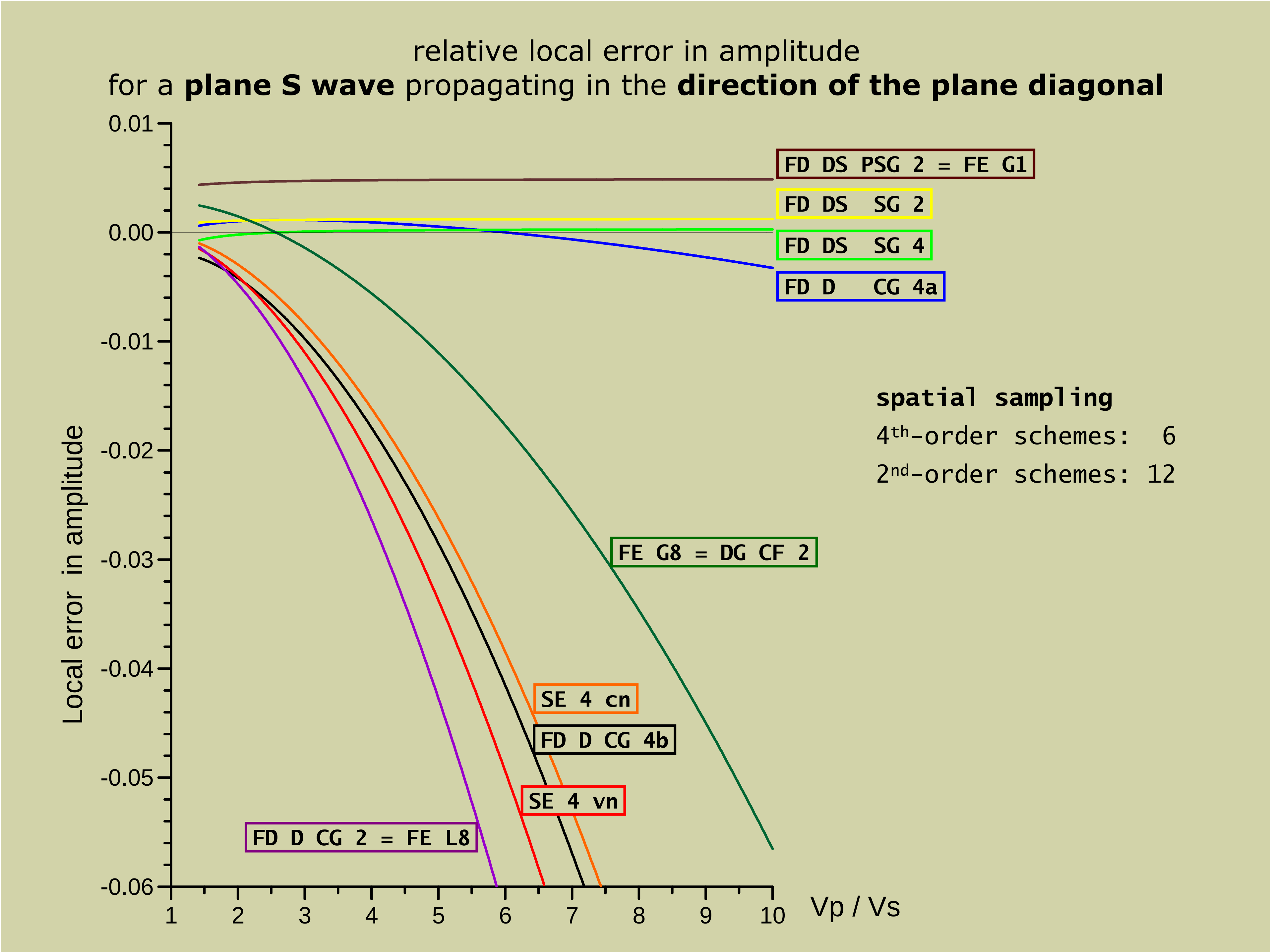
spatial sampling 2nd-order schemes: 12
VP / VS = 1.42
VP / VS = 5
VP / VS = 10
spatial sampling 4th-order schemes: 6

relative local error in amplitude
for a **plane S wave** propagating in the **direction of the plane diagonal**


relative local error in amplitude
for a **plane S wave** propagating in the **direction of the plane diagonal**
0.01
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
Local error in amplitude
1
2
3
4
5
6
7
8
9
10
Vp / Vs
FD DS PSG 2 = FE G1
FD DS SG 2
FD DS SG 4
FD D CG 4a
spatial sampling
4th-order schemes: 6
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look now at the **convergence** of the schemes

therefore,
consider

absolute value of the **average** local error in amplitude
for plane S waves propagating in **all directions**

$$\varepsilon = \left(\frac{\Delta t_{ref}}{\Delta t} \right)^2 \left| A_N - A_E \right|$$

absolute value of the **average** local error in amplitude
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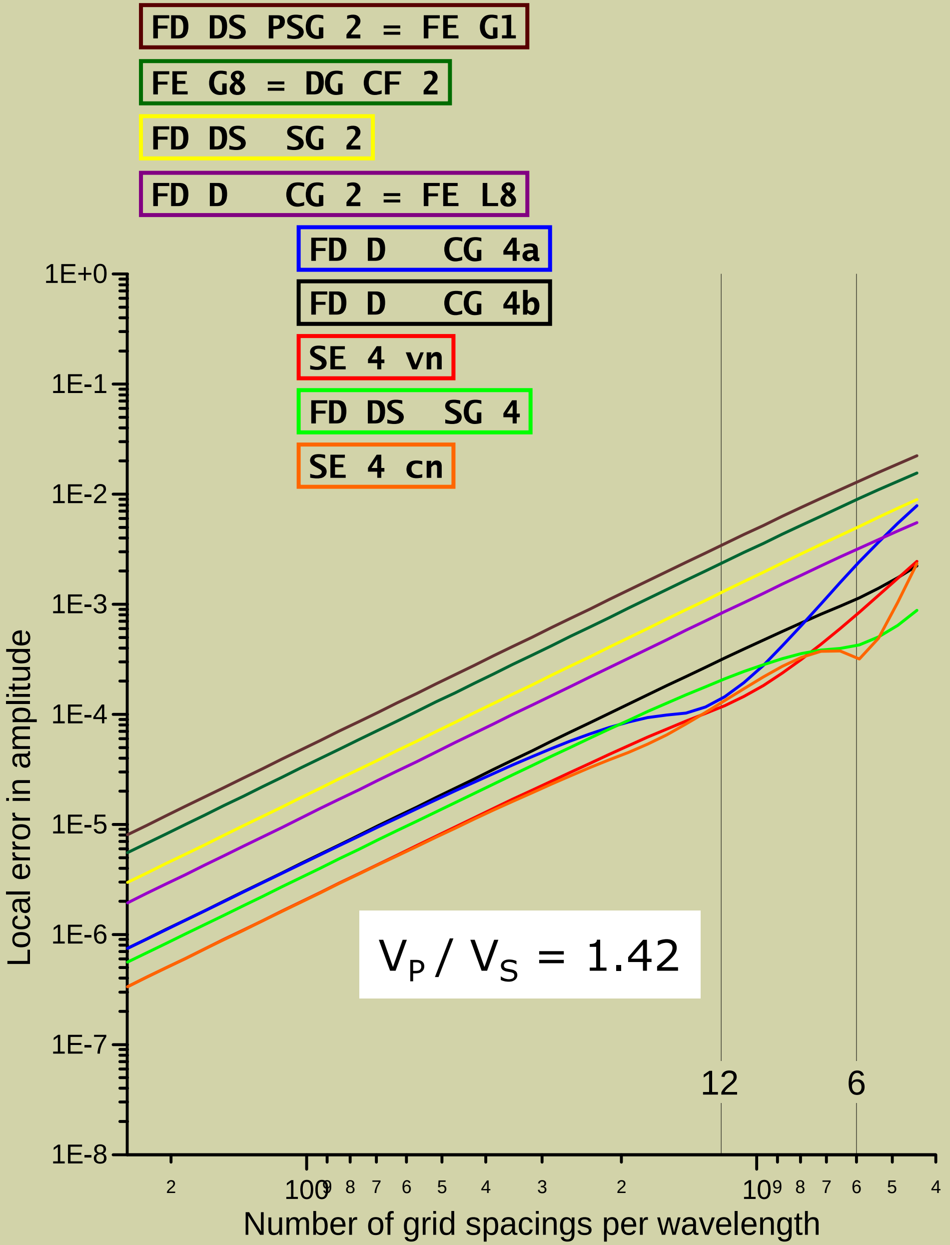

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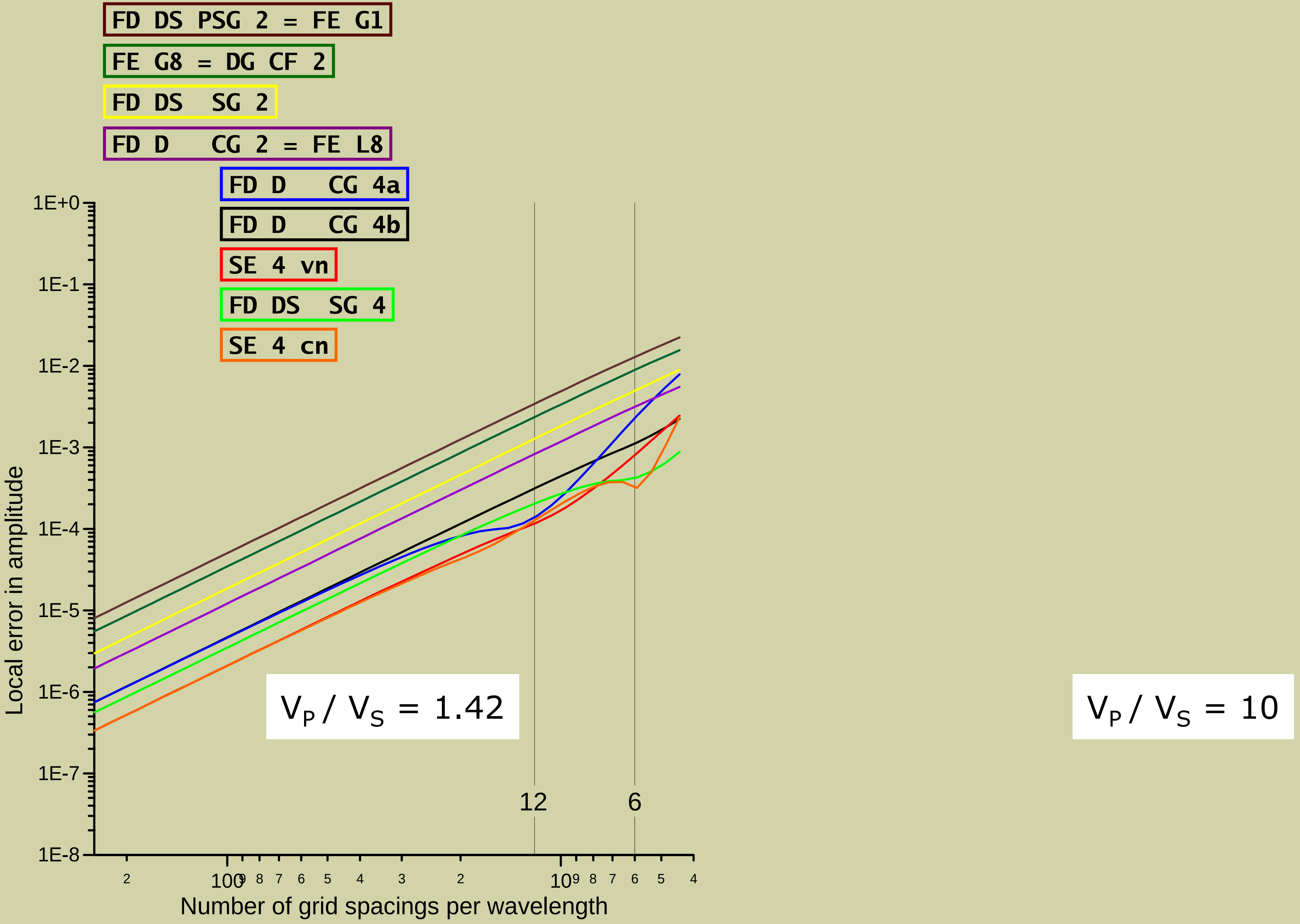

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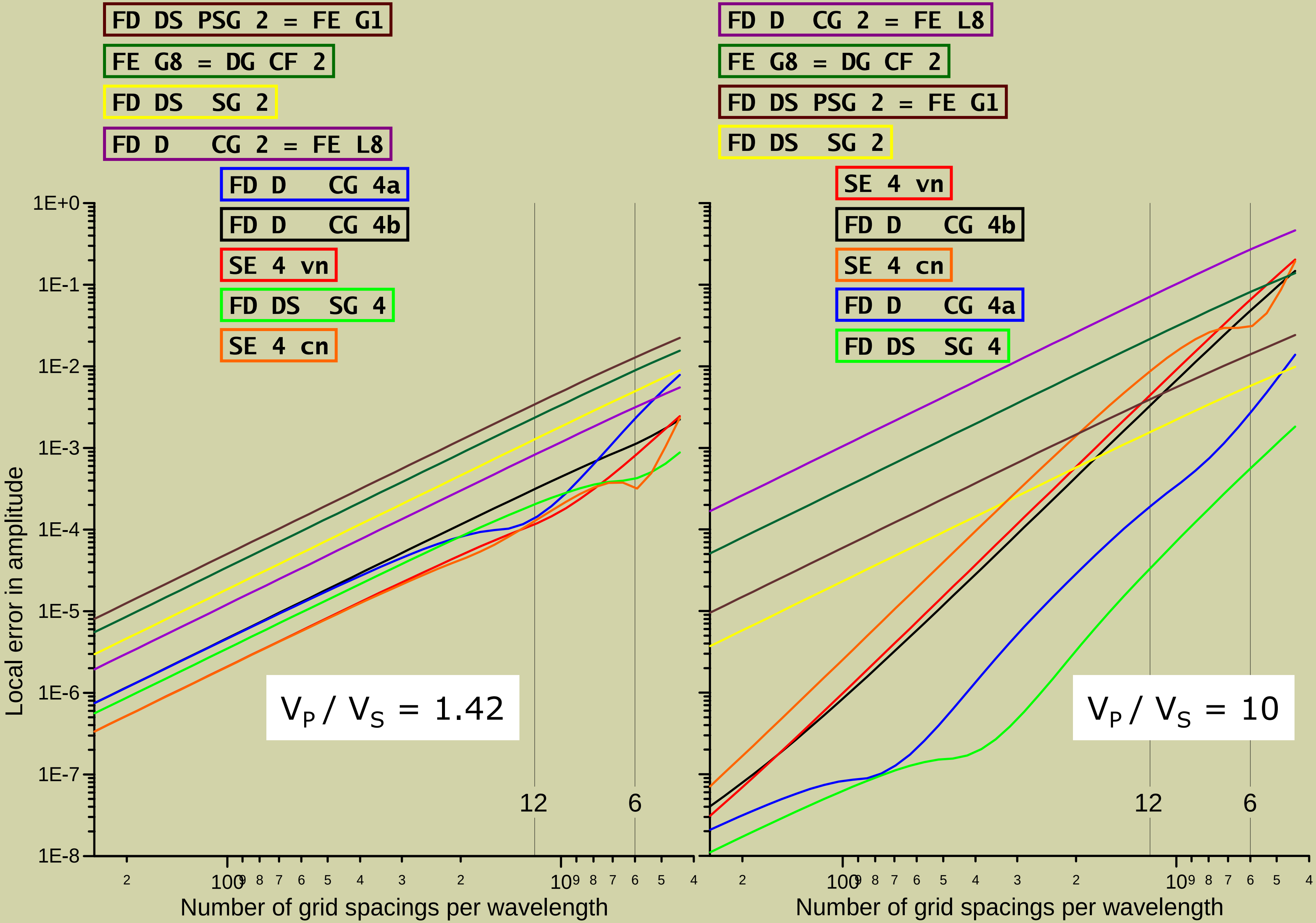

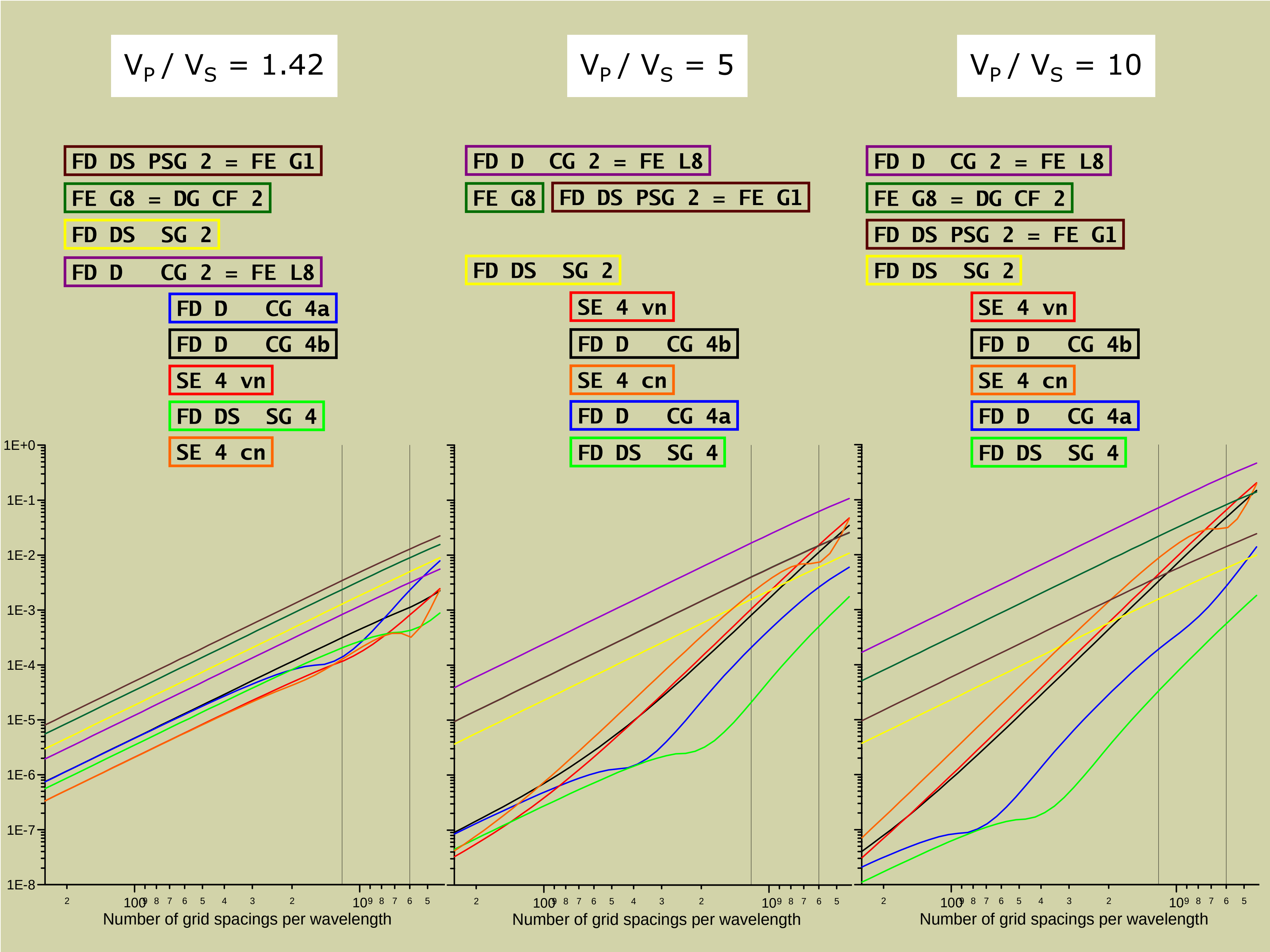

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1E+0
1E-1
1E-2
1E-3
1E-4
1E-5
1E-6
1E-7
1E-8
Number of grid spacings per wavelength

conclusions

we compared and analyzed 11 numerical schemes
for their behavior with a varying V_P / V_S ratio

conclusions

the inaccuracy of some schemes

with respect to the V_P / V_S ratio

inevitably leads to the

considerably lower computational efficiency

conclusions

the inaccuracy of some schemes

with respect to the V_P / V_S ratio

should be properly

accounted for

in the simulations for complex realistic structures paper on 2D schemes

Moczo, Kristek, Galis, Pazak

On accuracy
of the finite-difference and finite-element schemes
with respect to P-wave to S-wave speed ratio

Geophys. J. Int. 182, 493-510, 2010

available at www.nuquake.eu

thank you
for your attention