Non-periodic homogenization for elastic wave propagation in complex media

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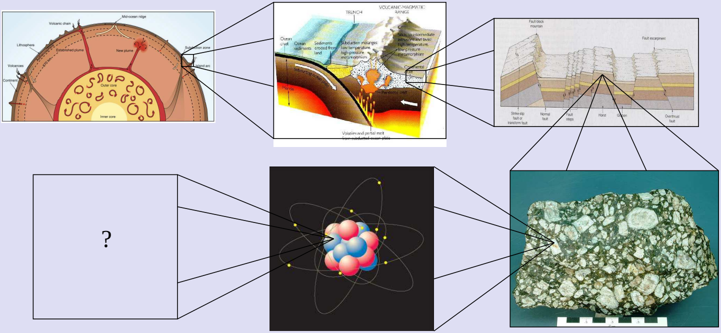
QUEST first workshop, 22/09/2010

Introduction and motivations

The earth contains inhomogeneities at all scales.



Scale separation

Observations indicate that waves of a given wavelength are sensitive to inhomogeneities of scales much smaller than this wavelength only in an effective way. This is one of the reasons why seismic waves can be used to image the earth.

Introduction and motivations

Inverse problem, full waveform inversion (FWI)

- FWI can only be performed in a limited frequency band.
- The best that can be recovered is what is "seen" by the wavefield. It is an effective version $(\rho^*, \mathbf{c}^*)$ of the real earth $(\rho, \mathbf{c})$.
- in that case, we know (at best) $(\rho^*, \mathbf{c}^*)$, but we wish to access to $(\rho, \mathbf{c})$.

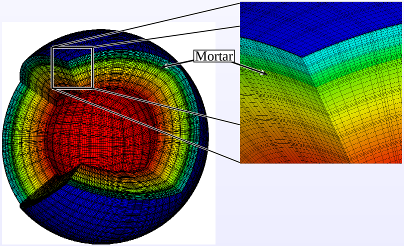
Forward modeling, full waveform modeling

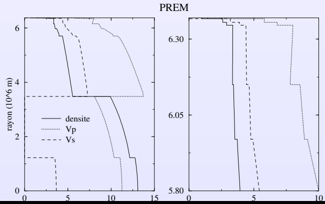
- full waveform modeling can only be performed in a limited frequency band.
- In many cases the elastic model $(\rho, \mathbf{c})$ we which to propagate in contains details much smaller than the minimum wavelength. The effective model $(\rho^*, \mathbf{c}^*)$ would be enough to propagate the wavefield.
- In that case, we know $(\rho, \mathbf{c})$, but we would like to have $(\rho^*, \mathbf{c}^*)$.

Small scales are an issue for the direct problem : e.g. SEM

Two simple but problematic SEM meshes :

Global scale





Regional scale

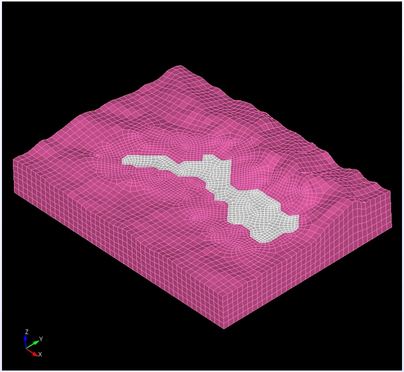


figure E. Delavaud

Introduction and motivations

In both cases (Full waveform modeling and inversion) knowing the relation between $(\rho, \mathbf{c})$ and $(\rho^*, \mathbf{c}^*)$ would be very useful, and we know since the work of Backus (1962) that the effective medium is not just a low pass filtered version of the original medium.

Backus (1962) shown that, form a finely layered media (1-D media), described by the A, C, F, L, N elastic parameters for TI media, the effective parameters A^*, C^*, F^*, L^*, N^* can be computed as :

$$\frac{1}{C^*} = \langle \frac{1}{C} \rangle \qquad A^* - \frac{F^{*2}}{C^*} = \langle A - \frac{F^2}{C} \rangle$$

$$\frac{1}{L^*} = \langle \frac{1}{L} \rangle \qquad \frac{F^*}{C^*} = \langle \frac{F}{C} \rangle$$

$$N^* = \langle N \rangle$$

For higher dimension (2-D, 3-D), the effect of small scales has been studied for long in mechanics with the two scale homogenization of periodic media (e.g. Auriault & Sanchez-Palencia (1977), Sanchez-Palencia (1980) ...) .

A simple periodic case : wave in a 1-D bar

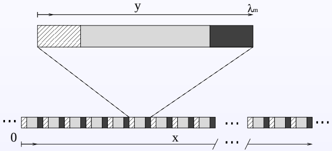
Assumptions :

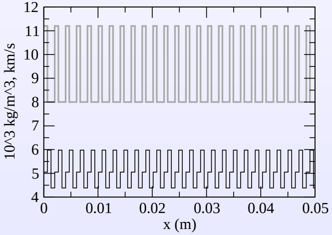
- λ_m : minimum wavelength
- E elastic modulus and ρ density ℓ periodic
- $\varepsilon = \frac{\ell}{\lambda_m} << 1$

Wave equation for the set of problems (parametrized by ε) :

$$\rho^\varepsilon \partial_{tt} u^\varepsilon - \partial_x \sigma^\varepsilon = f^\varepsilon$$

$$\sigma^\varepsilon = E^\varepsilon \partial_x u^\varepsilon$$





Classical two scale expansion

1. Introduction of the fast variable $y = \frac{x}{\varepsilon}$
2. Introduction of $\rho(y) = \rho^\varepsilon(\varepsilon y)$ and $E(y) = E^\varepsilon(\varepsilon y)$
3. As $\varepsilon \longrightarrow 0$ y and x are treated as independent variables implying $\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}$
4. Solutions thought as

$$u^\varepsilon(x,t) = \sum_{i\geq 0} \varepsilon^i u^i(x, y = \frac{x}{\varepsilon}, t)$$

$$\sigma^\varepsilon(x,t) = \sum_{i\geq -1} \varepsilon^i \sigma^i(x, y = \frac{x}{\varepsilon}, t)$$

5. Series of equation to be solved for each i :

$$\rho\partial_{tt}u^i + \partial_x\sigma^i + \partial_y\sigma^{i+1} = f^i$$

$$\sigma^i = E(\partial_x u^i + \partial_y u^{i+1})$$

Classical two scale expansion : resolution

Cell average :

$$\langle h\rangle(x) = \frac{1}{\lambda_m}\int_0^{\lambda_m} h(x,y)dy$$

Solving the asymptotic equations one by one gives

1. $\sigma^{-1} = 0$, $u^0 = \langle u^0\rangle$, $\sigma^0 = \langle \sigma^0\rangle$
2. $u^1(x,y) = \chi^1(y)\partial_x u^0(x) + \langle u^1\rangle(x)$
   $\chi^1(y)$ periodic with $\langle \chi^1\rangle = 0$ solution of cell problem

$$\partial_y\left[E(1+\partial_y\chi^1)\right] = 0$$

3. u^0 and σ^0 of the order 0 effective wave equation

$$\langle\rho\rangle\partial_{tt}u^0 - \partial_x\sigma^0 = f^0 \qquad \sigma^0 = E^*\partial_x u^0$$

   with $E^* = \langle E(1+\partial_y\chi^1)\rangle$
4. analytical solution of the cell problem gives : $E^* = \langle \frac{1}{E}\rangle^{-1}$

Classical two scale expansion : summary

We have found that :

$$u^\varepsilon = \langle u^0 \rangle + \varepsilon \left(\langle u^1 \rangle + \chi^1 \partial_x \langle u^0 \rangle \right) + \ldots$$
$$\sigma^\varepsilon = \langle \sigma^0 \rangle + \varepsilon \left(\langle \sigma^1 \rangle + \chi^1_\sigma \partial_x \langle \sigma^0 \rangle \right) + \ldots$$

χ^1 is the first order periodic corrector with $\langle \chi^1 \rangle = 0$ solution of cell problem

$$\partial \left[E(1 + \partial_y \chi^1) \right] = 0$$

$u = \langle u^0 \rangle + \varepsilon \langle u^1 \rangle$, $\sigma = \langle \sigma^0 \rangle + \varepsilon \langle \sigma^1 \rangle$ are solution of the effective equation (here at the order 1) :

$$\langle \rho \rangle \partial_{tt} u - \partial_x \sigma = f \, , \qquad \sigma = E^* \partial_x u$$

with $E^* = \langle E(1 + \partial_y \chi^1) \rangle$ is the effective elastic parameter.
The final solution can be obtained with

$$u^\varepsilon = u(1 + \varepsilon \chi^1 \partial_x) + O(\varepsilon^2)$$

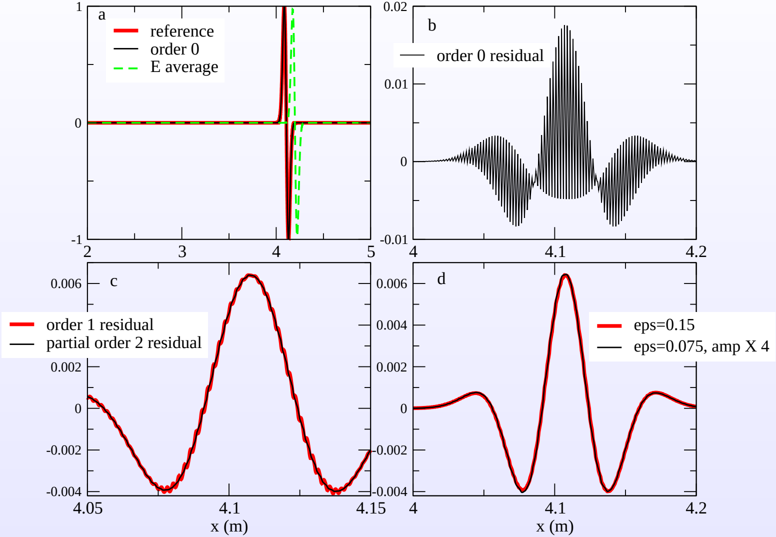
$$\epsilon^\varepsilon = \epsilon(1 + \partial_y \chi^1) + O(\varepsilon)$$

Classical two scale expansion : summary

Note that,

- in this simple 1D case, there is an analytical solution to the cell problem leading to $1/E^* = \langle 1/E \rangle$. There is not such an analytical solution for higher dimensions.
- at the order 0 the solutions do not depend on the microscopic scale. This is still true for higher dimension for u^0 but not for σ^0 ;
- order > 0 : the boundary condition changes (e.g. Neumann condition becomes DtN) ;
- order > 1 : the effective equation changes (it is not a classical wave equation anymore) ;

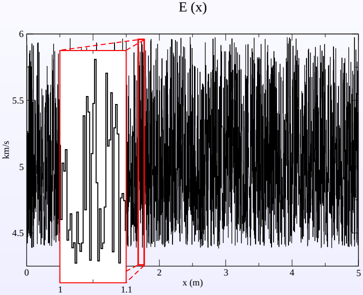
An example



"E average" means computed with $E^* = \langle E \rangle$

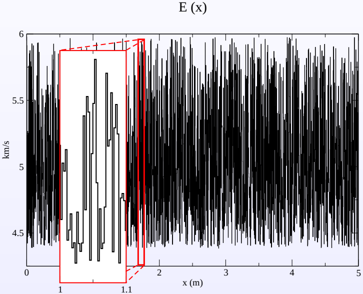
Non-periodic case

What can be done in the non periodic case ?



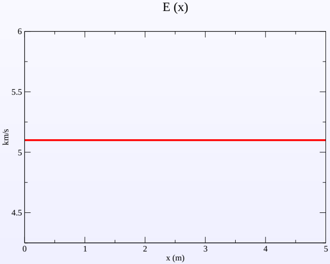
Non-periodic case

What can be done in the non periodic case ?



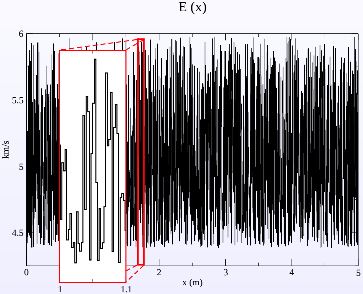
periodic homogenization on the whole domain ?

$\Longrightarrow$



Non-periodic case

What can be done in the non periodic case ?



periodic homogenization on the whole domain ?

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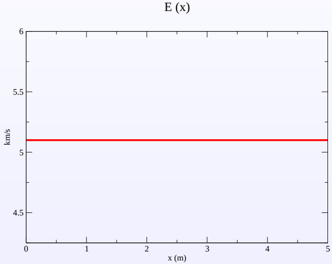
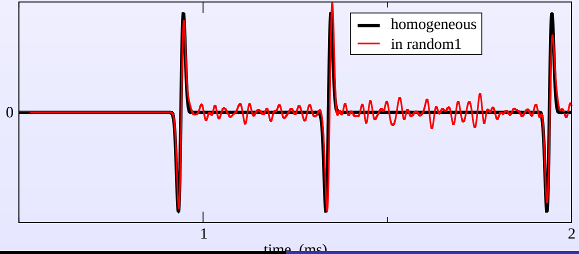
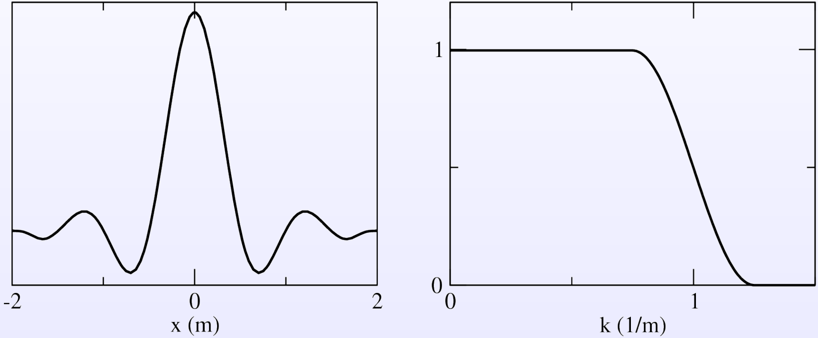




Non-periodic case : an intuitive solution

A different spatial filter :$\mathcal{F}^{\varepsilon_0}(h)(x) = \int h(x')w_{\varepsilon_0}(x-x')dx'$

An example of wavelet w_{ε_0} :

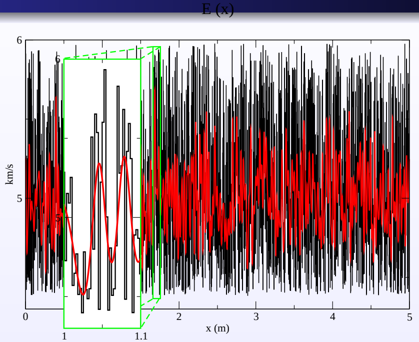


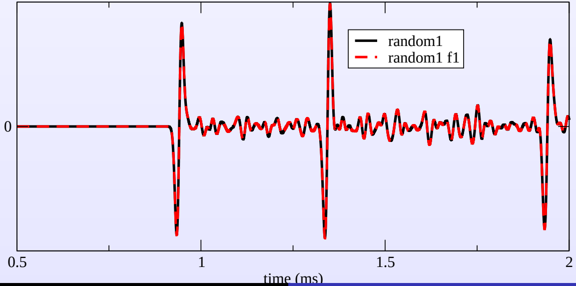
Intuitive order 0 solution, using :

$$\frac{1}{E^*}(x) = \mathcal{F}^{\varepsilon_0}\left(\frac{1}{E}\right)(x)$$

$$\varepsilon_0 = \frac{\lambda_{cutoff}}{\lambda_{min}}$$

Testing this intuitive idea





From the intuitive solution to something more general

- Replacing $< \; . \; >$ by $\mathcal{F}^{\varepsilon_0}(\; . \;)$ in the formulas obtained in the periodic case doesn't give the proper results
- a good solution is to build $E(x, y)$ such

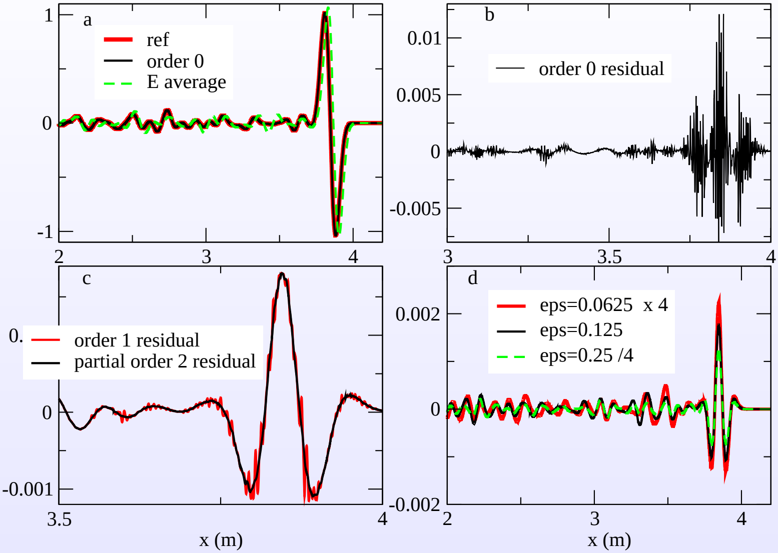
$$\frac{1}{E^{\varepsilon_0}}(x,y) = \mathcal{F}^{\varepsilon_0}\left(\frac{1}{E}\right)(x) + \left(\frac{1}{E} - \mathcal{F}^{\varepsilon_0}\left(\frac{1}{E}\right)\right)(y)$$

  Doing so, we can show that

$$\frac{1}{E^{\varepsilon_0 *}} = \langle \frac{1}{E^{\varepsilon_0}} \rangle = \mathcal{F}^{\varepsilon_0}\left(\frac{1}{E}\right)$$

- Knowing $E^{\varepsilon_0}(x, y)$, we can follow the same process as for the periodic case

An non-periodic example



"E average" means computed with $E^* = \mathcal{F}^{\varepsilon 0}(E)$

From 1D to 2D/3D ?

2D PSV case as an example

$$\rho \ddot{\mathbf{u}} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}$$

$$\boldsymbol{\sigma} = \mathbf{c}^0 : \nabla \mathbf{u}$$

$$\mathbf{c}^0(\mathbf{x}) = \{c^0_{ijkl}(\mathbf{x})\}, \ (i,j,k,l) \in \{1,2\},$$

$$c^0_{ijkl} = c^0_{jikl} = c^0_{ijlk} = c^0_{klij} \,.$$

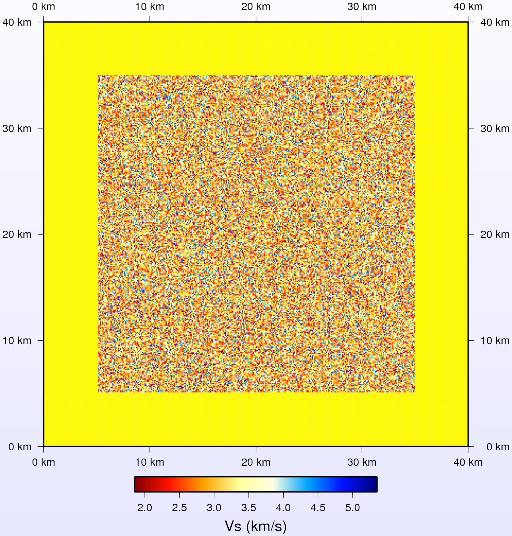
$$\mathbf{u} = {}^t(u_1, u_2)$$

2D/3D non-periodic homogenization

For 2D and 3D, the homogenization follows a similar procedure to the 1D case. This 2D/3D process is well known for the periodic case. Nevertheless, the extension to non-periodic media is more difficult than for the 1D case.
Difficulty : There is no analytical solutions for $\mathbf{c}^\star$ and therefore there is obvious way to construct $\mathbf{c}(\mathbf{x}, \mathbf{y})$ in the non periodic case.

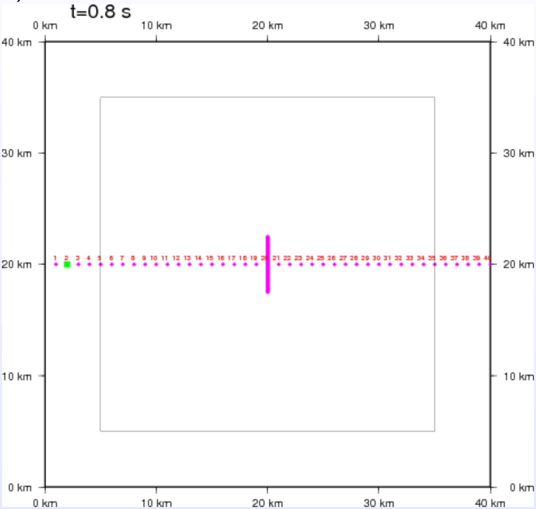
A 2-D random example

The inner square is a 300×300 homogeneous squares cells (100×100 m^2). In each cell, the density, λ and μ are determine randomly with contrast up to 50%.

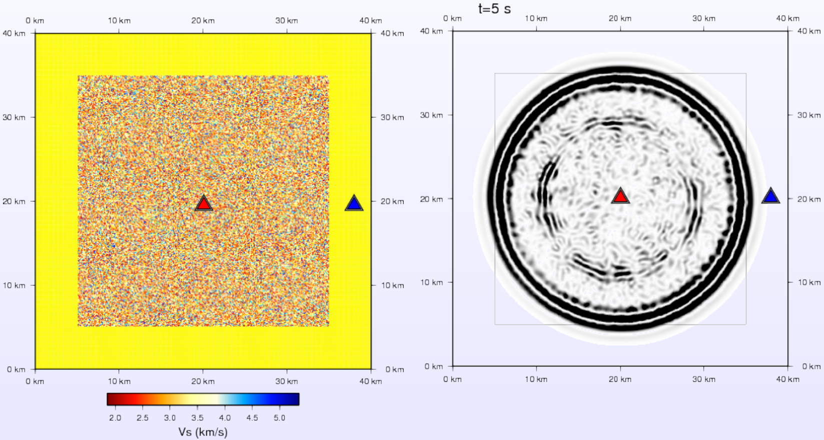


A 2-D random example

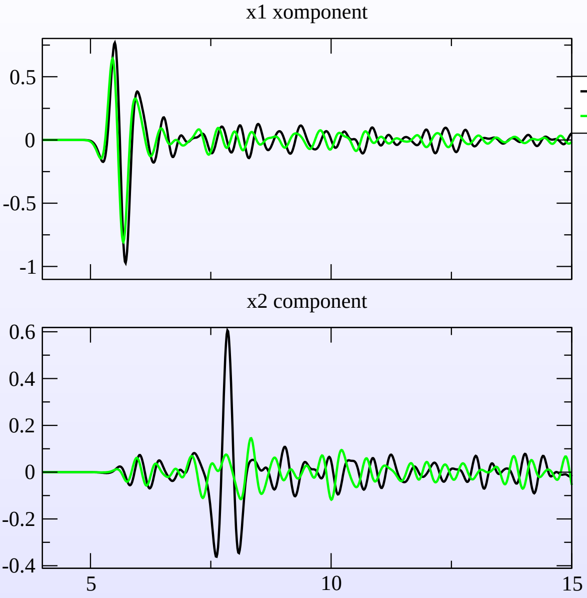
The source is an explosion (1.5Hz of central frequency, 3.6Hz. $\lambda_{min} \simeq 0.8km$.)

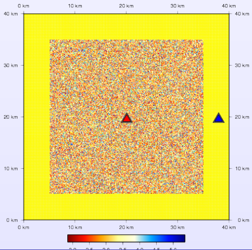


A 2-D random example : source at the center of the square



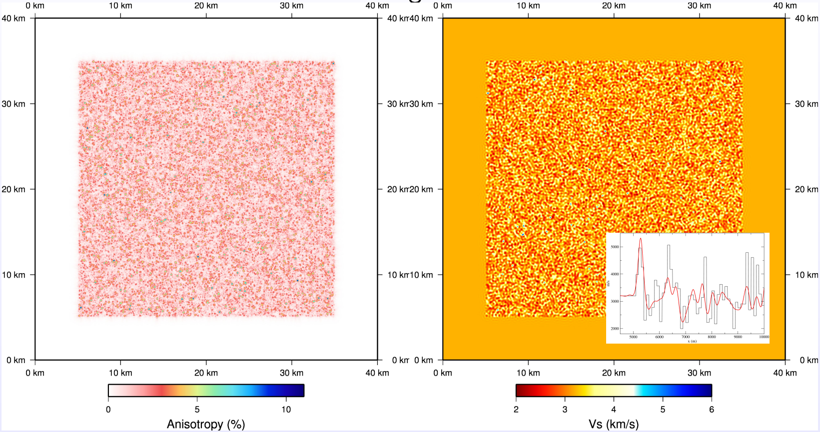
A 2-D random example : source at the center of the square



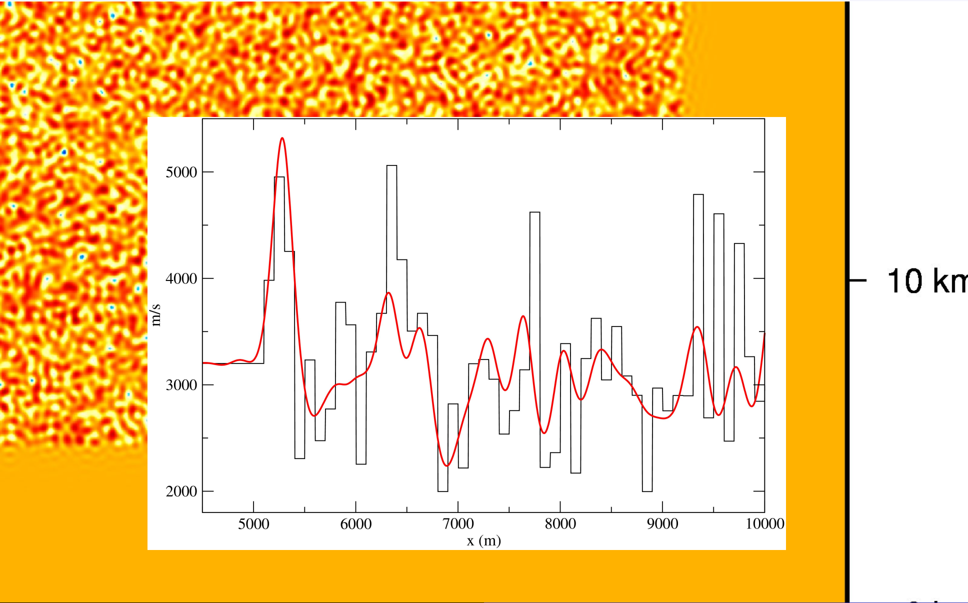


A 2-D random example

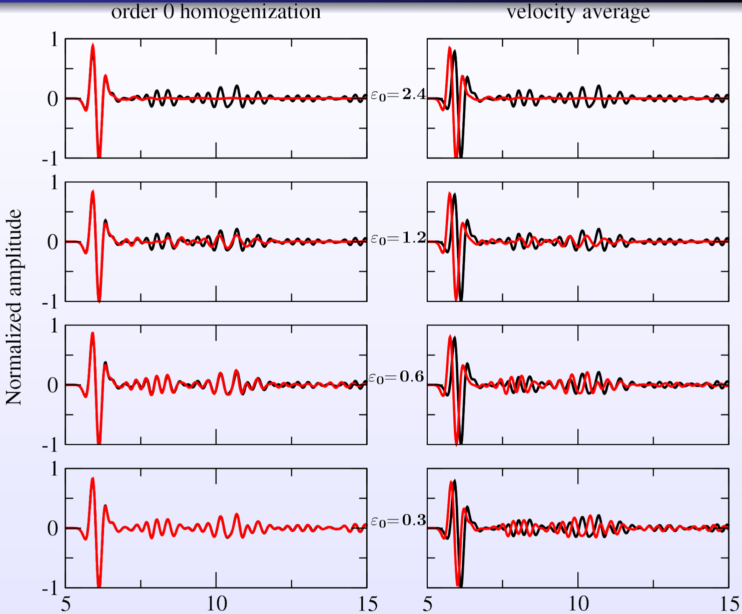
Order 0 homogenized model



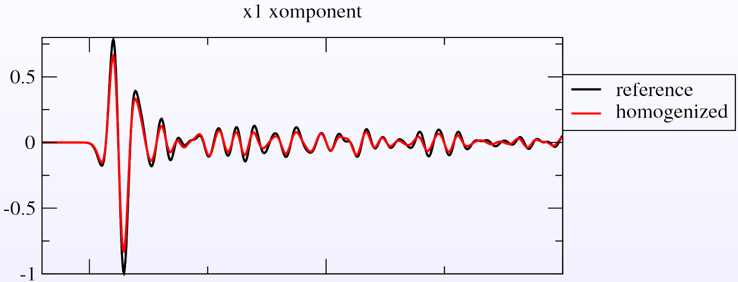
A 2-D random example

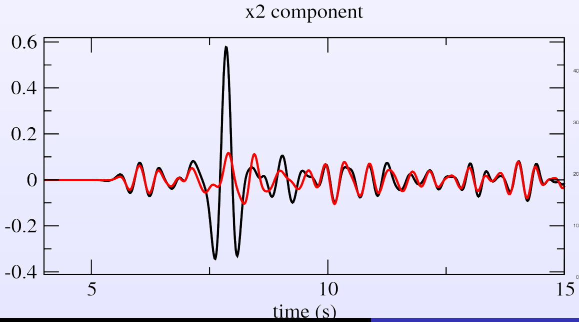


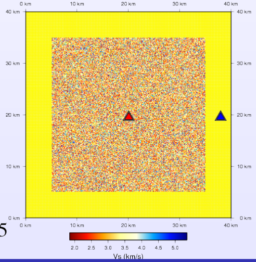
A 2-D random example



A 2-D random example : source at the center of the square







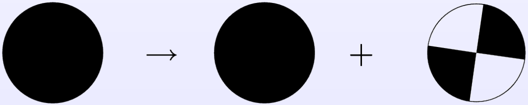
A 2-D random example : source at the center of the square

The homogenization theory at the order 0 allows to correct the moment tensor for interaction with inhomogeneities within the near-field of the source :

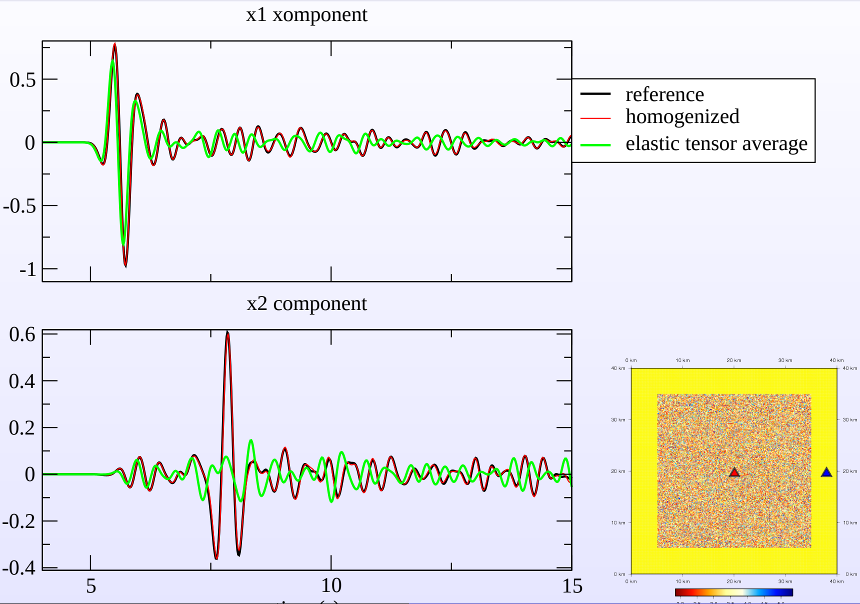
$$\mathbf{M}^* = \mathbf{G}(\mathbf{x}_0) : \mathbf{M}$$

For our example :

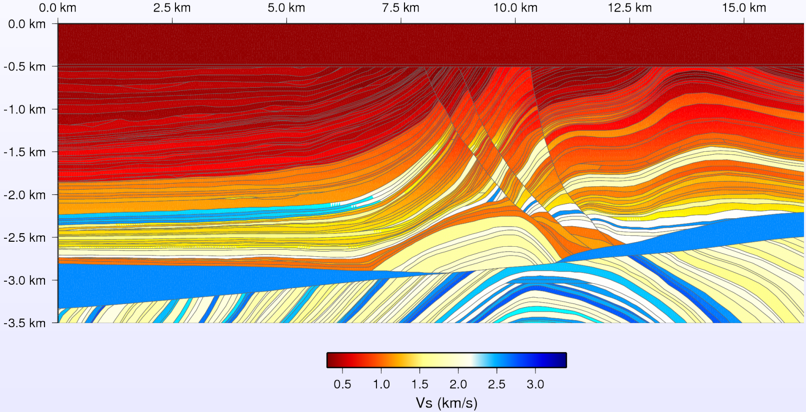
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1.25 & 0 \\ 0 & 1.25 \end{pmatrix} \quad \begin{pmatrix} -0.05 & -0.17 \\ -0.17 & +0.05 \end{pmatrix}$$



A 2-D random example : source at the center of the square



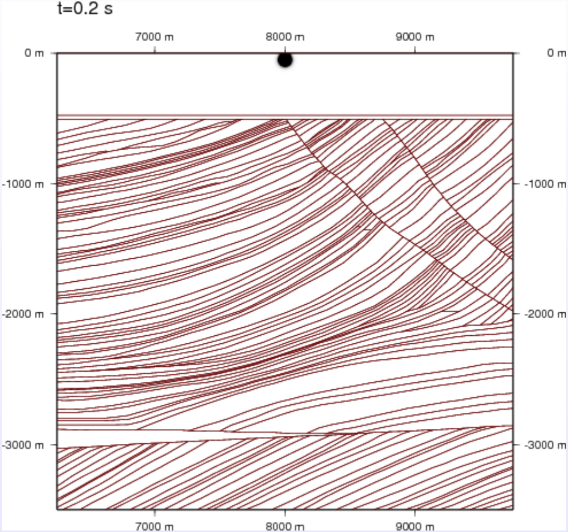
Maroumsi2 example



Marmousi2 example

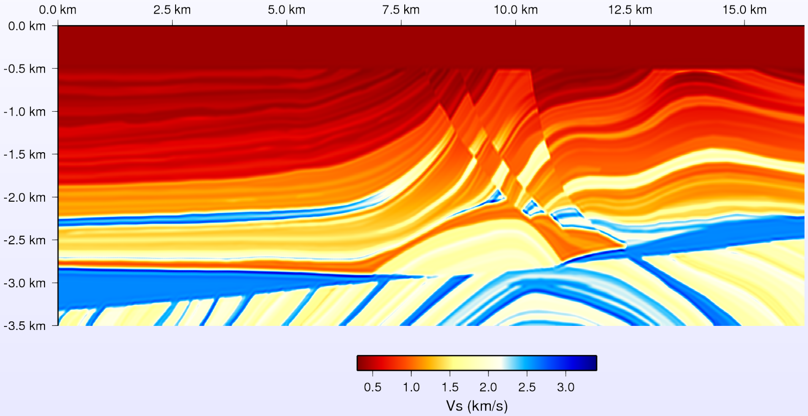
Reference solution :

-1 week computation on 64 CPU.
-Source : explosion
-central frequency : 6Hz (15Hz corner)



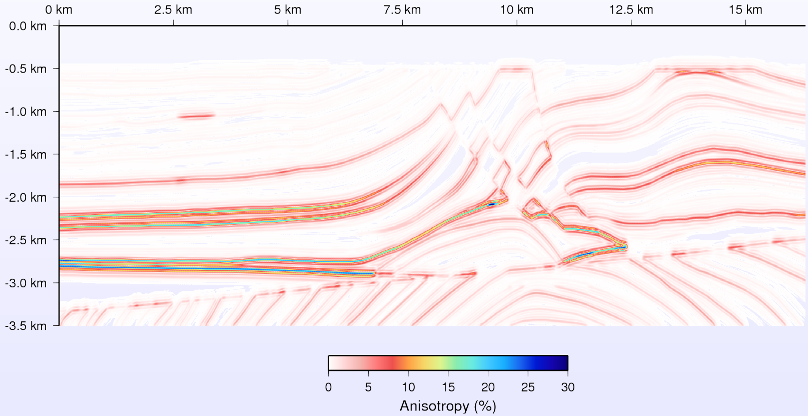
Marmousi2 example

Order 0 homogenized model (ε varies from 3 at the top of the model to 0.25 at the bottom)



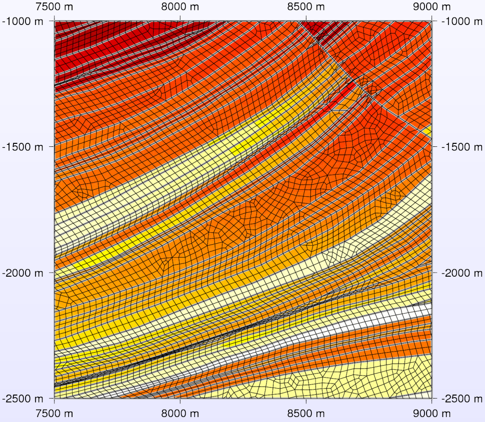
Marmousi2 example

Order 0 homogenized model

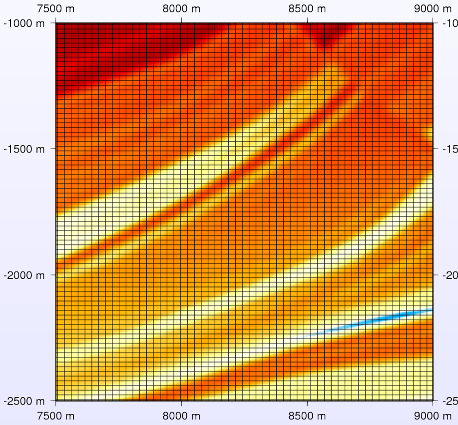


Marmousi2 example

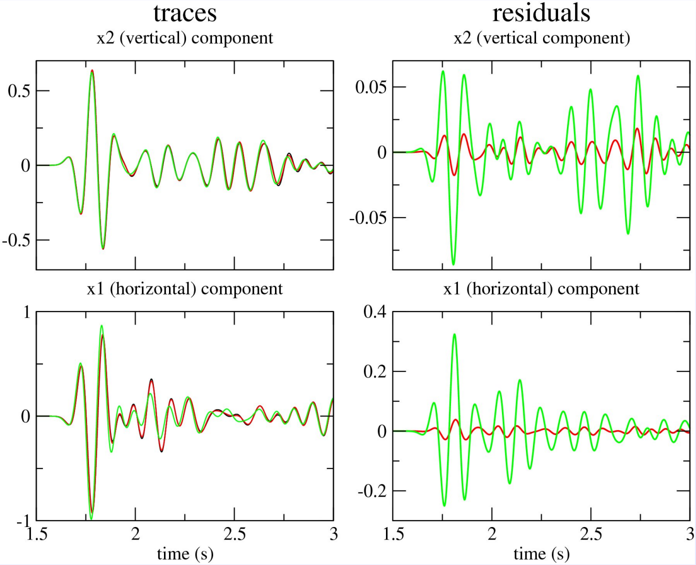
Mesh for the original model



Mesh for the order 0 homogenized model



Marmousi2 example



Conclusions and perspectives

- we have shown a homogenization process for the wave equation in the non-periodic case
- this process can be extended form 1-D to 2-D/3-D
- many issues remain to be solved like boundary conditions in non-periodic 2-D/3-D cases (1-D case have been solved)
- This should be useful for both forward and inverse problems

Capdeville, Y. Guillot, L. and J. J. Marigo (2010) 2-D non periodic homogenization to upscale elastic media for P-SV waves. *Geophys. J. Int.*, 182, 903-922

Guillot, L., Capdeville, Y. and J. J. Marigo (2010) 2-D non periodic homogenization for the SH wave equation *Geophys. J. Int.*, 182, 1438-1454

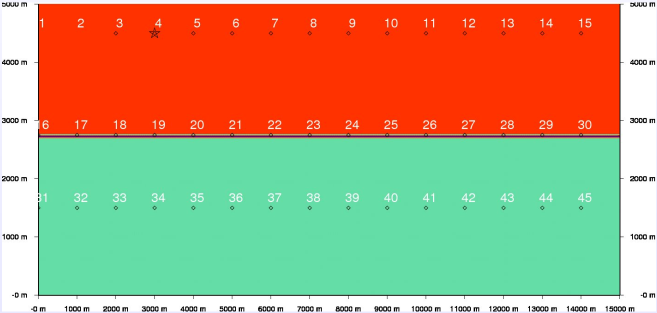
Capdeville, Y. Guillot, L. and J. J. Marigo (2010) 1-D non periodic homogenization for the 1wave equation *Geophys. J. Int.*, 181, 897-910

Simple examples. Case A

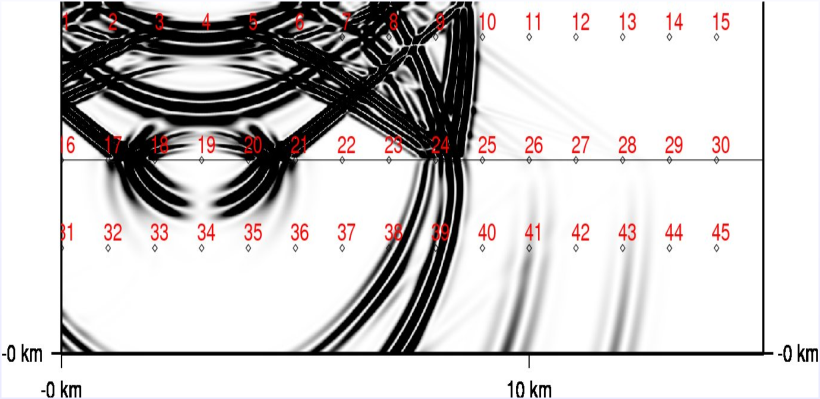
Top :
Vp=2.4km/s
Vs=1.2km/s
ρ=1500kg/m^3

Bottom :
Vp=5.6km/s
Vs=2.8km/s
ρ=2800kg/m^3

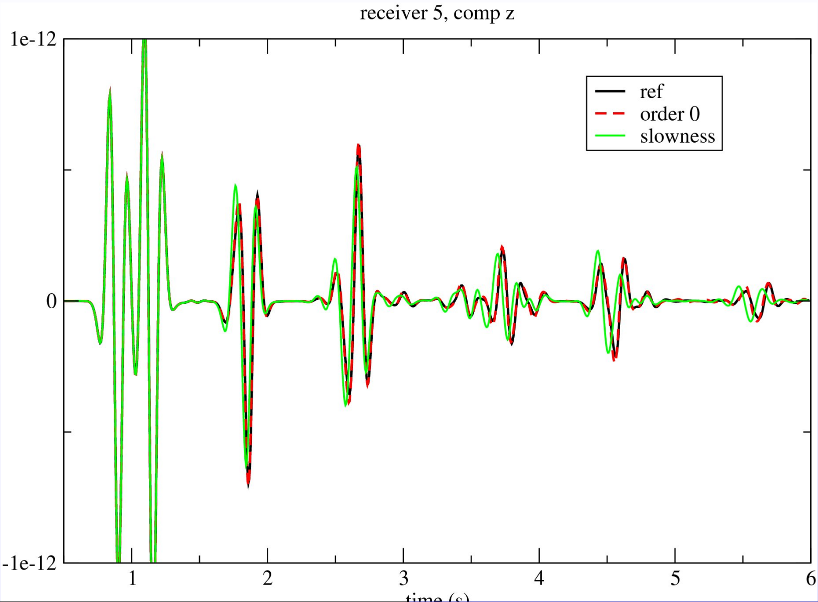
f_{max} =12.5Hz
λ_{min} =100m
ε =0.5



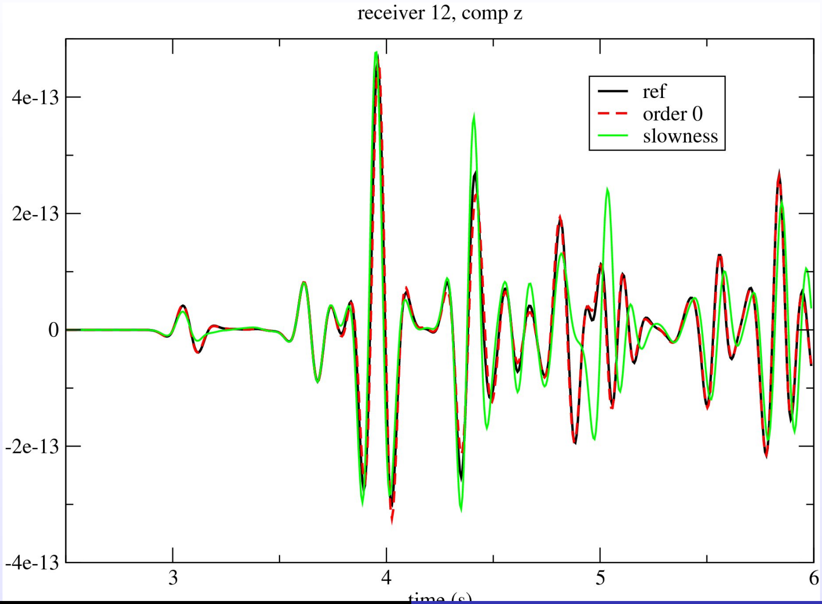
Energy snapshot at t=2.6s



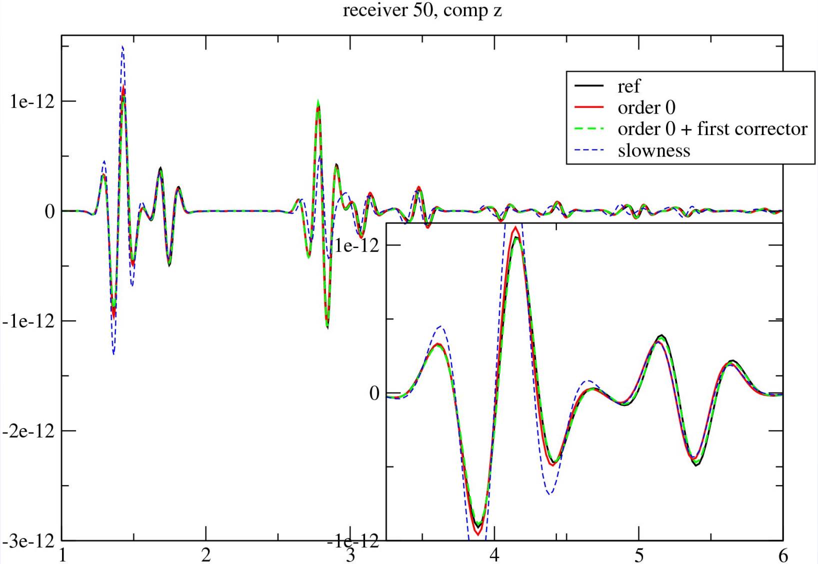
Small offset trace (receiver 5)



Large offset trace (receiver 12)



trace with first order corrector effect (receiver 50)

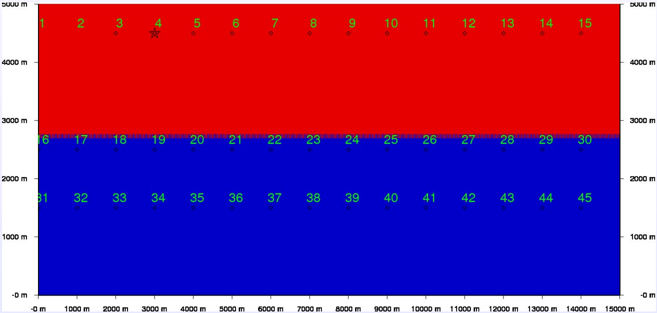


Simple examples. Case B

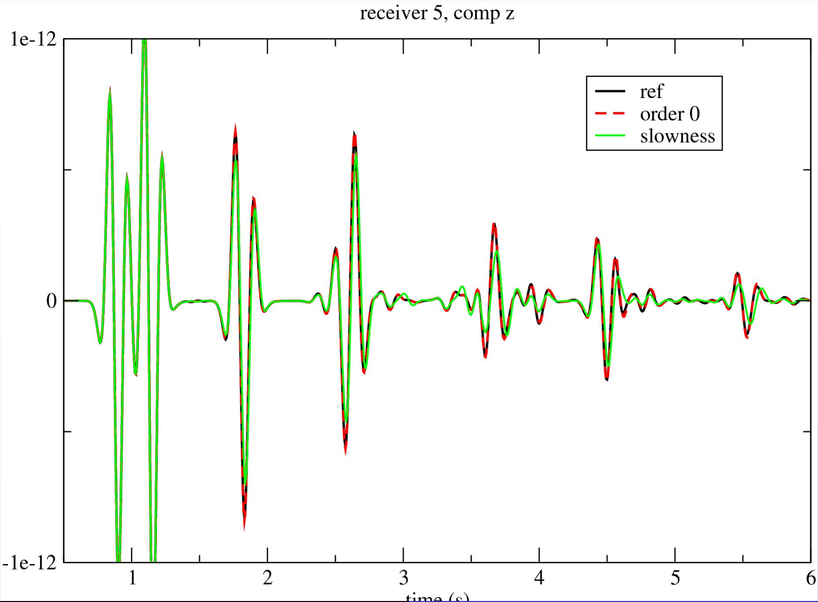
Top :
Vp=2.4km/s
Vs=1.2km/s
ρ=1500kg/m^3

Bottom :
Vp=5.6km/s
Vs=2.8km/s
ρ=2800kg/m^3

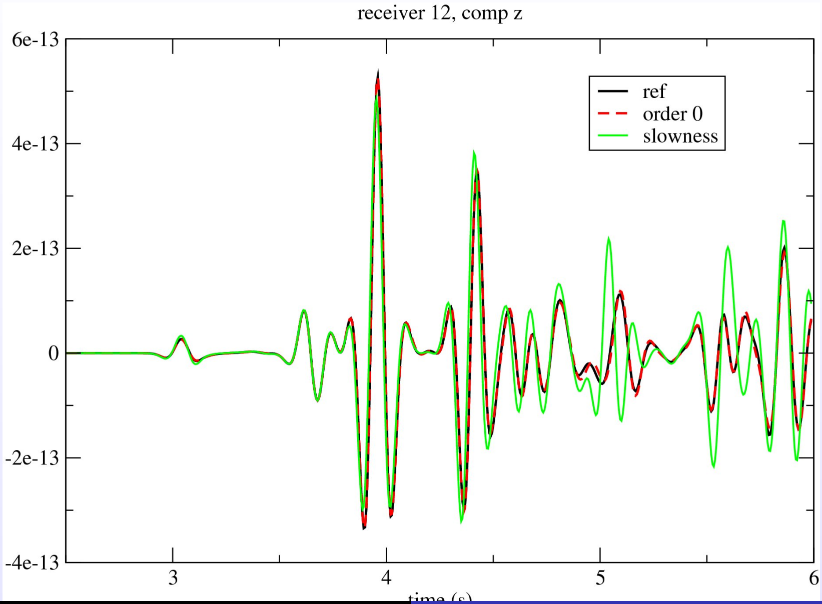
f_{max} =12.5Hz
λ_{min} =100m
ε =0.5



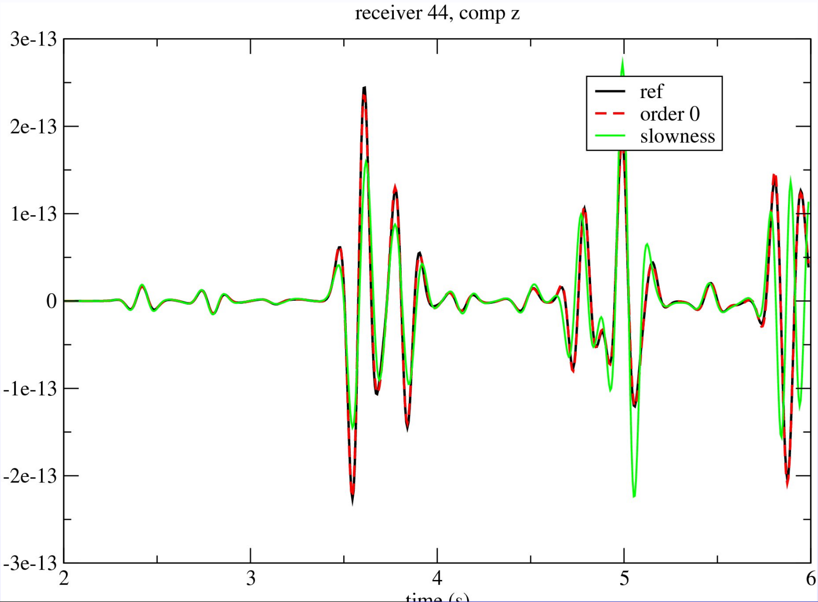
Small offset trace (receiver 5)



Large offset trace (receiver 12)

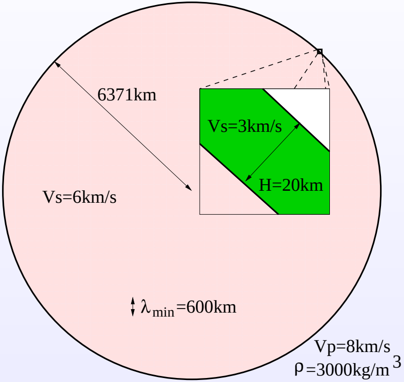


Large offset, transmission, trace (receiver 42)

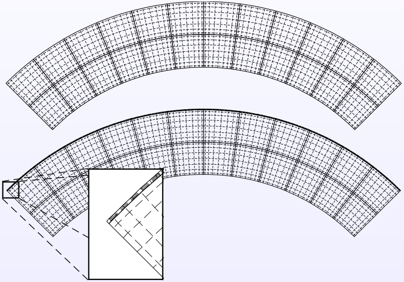


A limited aspect of this general problem : thin shallow layers

An homogeneous sphere with a thin slow layer on the top



Mesh 1 (no shallow layer)



Mesh 2 (with shallow layer)

Source corner period : 100 s

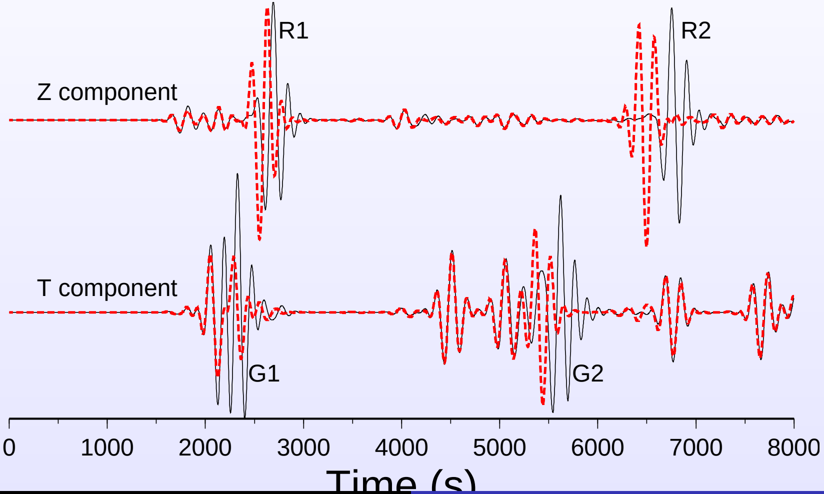
Just because of the stability condition, propagating in mesh 2 is 30 times more computing intensive than in mesh 1.

This is a generic case of crustal models (e.g. 3SMAC, CRUST2.0) implementation in SEM at the global scale.

Two classical solutions

Solution 1 : Ignoring the shallow layer

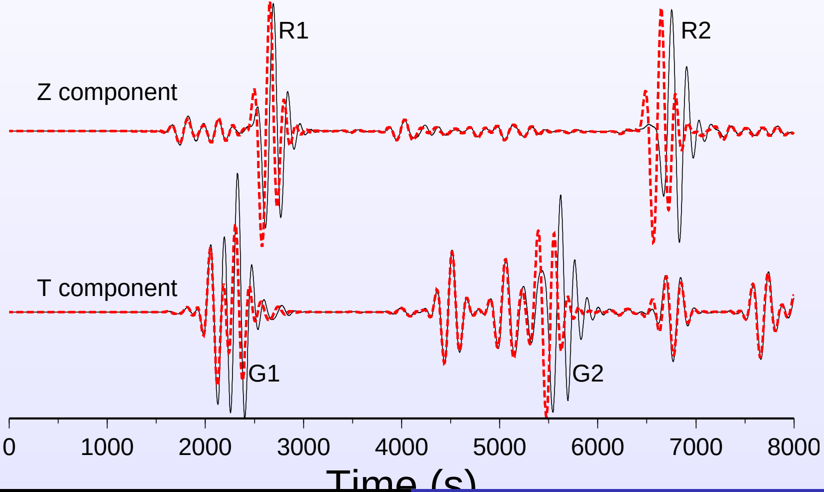
Black line : reference solution ; red line : approximate solution



Two classical solutions

Solution 2 : the discontinuity is not honored by the mesh

Black line : reference solution ; red line : approximate solution



A third solution : matching asymptotic expansions

Assumption : $\varepsilon = \frac{H}{\lambda_{min}} << 1$

New variable : $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$

expansion in the shallow layer :

$$\mathbf{u}_c^\varepsilon(\mathbf{y}) = \sum_i \varepsilon^i \mathbf{u}_c^i(\mathbf{y})$$

$$\boldsymbol{\sigma}_c^\varepsilon(\mathbf{y}) = \sum_i \varepsilon^i \boldsymbol{\sigma}_c^i(\mathbf{y})$$

$$\rho \ddot{\mathbf{u}}_c^\varepsilon - \nabla \cdot \boldsymbol{\sigma}_c^\varepsilon = \mathbf{f}$$

$$\boldsymbol{\sigma}_c^\varepsilon = \mathbf{c} : \epsilon(\mathbf{u}_c^\varepsilon)$$

$\mathbf{c}(\mathbf{y}) = \mathbf{c}^\varepsilon(\mathbf{x}/\varepsilon)$

Boundary condition : free surface

expansion in the volume :

$$\mathbf{u}^\varepsilon(\mathbf{x}) = \sum_i \varepsilon^i \mathbf{u}^i(\mathbf{x})$$

$$\boldsymbol{\sigma}^\varepsilon(\mathbf{x}) = \sum_i \varepsilon^i \boldsymbol{\sigma}^i(\mathbf{x})$$

$$\rho^s \ddot{\mathbf{u}}^\varepsilon - \nabla \cdot \boldsymbol{\sigma}^\varepsilon = \mathbf{f}$$

$$\boldsymbol{\sigma}^\varepsilon = \mathbf{c}^s : \epsilon(\mathbf{u}^\varepsilon)$$

$\mathbf{c}^s(\mathbf{x})$ is “smooth”

Boundary condition : regularity at the center of the earth

the solutions must match in region where both solutions are valid

- introducing the expansion in the waves equation
- using $\frac{\partial}{\partial \mathbf{x}} \rightarrow \frac{1}{\varepsilon}\frac{\partial}{\partial \mathbf{y}}$
- identifying terms in ε^i

a series of equations is obtained that can be solved one by one

- order 0 : regular wave equation with free boundary condition in the "smooth" model ($\mathbf{c}^s$)
- order i ($i > 0$) : regular wave equation in the "smooth" model ($\mathbf{c}^s$) but with a special DtN boundary condition.

At the order 2, on the surface :

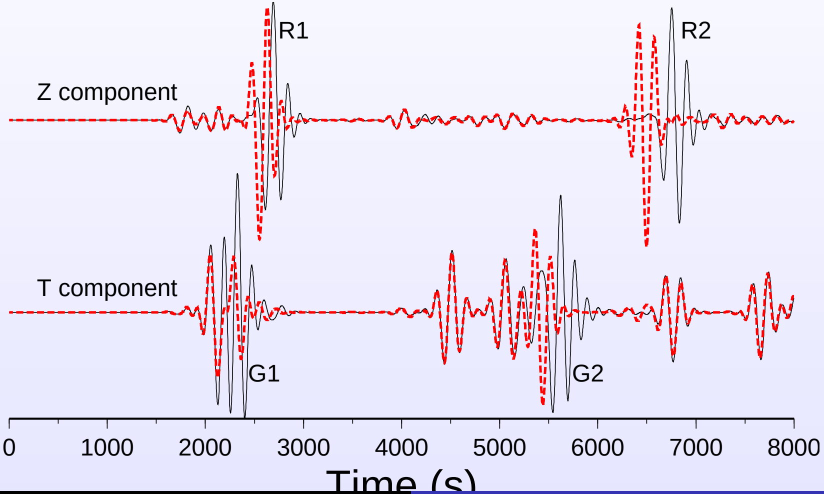
$$\mathbf{t} = \varepsilon \left\{ \; X^1_\rho \ddot{\mathbf{u}} - X^1_{a1} \; \boldsymbol{\nabla}_1 (\boldsymbol{\nabla}_1 \cdot \mathbf{u}_1) + X^1_N \; \boldsymbol{\nabla}_1 \times \boldsymbol{\nabla}_1 \times \mathbf{u}_1 \; \right\}$$
$$+ \varepsilon^2 \left\{ \; X^2_{a1} \left(\nabla^2_1 (\boldsymbol{\nabla}_1 \cdot \mathbf{u}_1) \, \hat{\mathbf{z}} - \boldsymbol{\nabla}_1 \mathrm{u}_z \; \right) + X^2_b \; \left((\boldsymbol{\nabla}_1 \cdot \ddot{\mathbf{u}}_1) \, \hat{\mathbf{z}} - \boldsymbol{\nabla}_1 \ddot{\mathrm{u}}_z \right) \; \right\}$$

$$\text{with (e.g.)} \qquad X^1_\rho = - \int_0^{\frac{H}{\varepsilon}} \left(\rho(y) - \rho^s(a) \right) dy$$

Order 0 matching asymptotic result

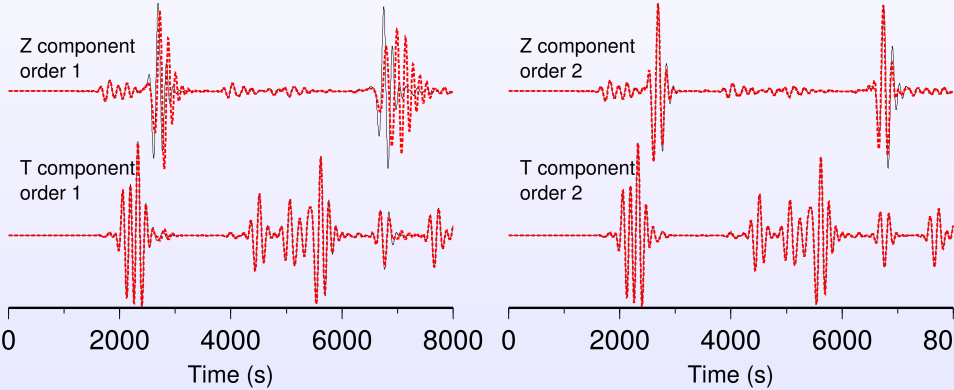
Same as "ignoring the shallow layer" solution

Black line : reference solution ; red line : approximate solution



Order two matching asymptotic results

Black line : reference solution ; red line : order 1 or 2 solution



Conclusions on matching asymptotic expansions for shallow layers

Interest

It gives a macroscopic view of small scale just bellow the surface. It is useful for

- forward modeling technique (e.g accurate crust model implementation) ;
- seismic imaging technique (e.g. can solve the global scale crustal correction issues).

Limitations

- the frequency band of accuracy is fixed by the thickness of the shallow layer ;
- the DtN can lead to instabilities (but this can be worked out) ;
- it doesn't solve the problem of deep small scales.

To move to a more general case, we need two scale homogenization