a short lecture on

# The spectral-element method

by

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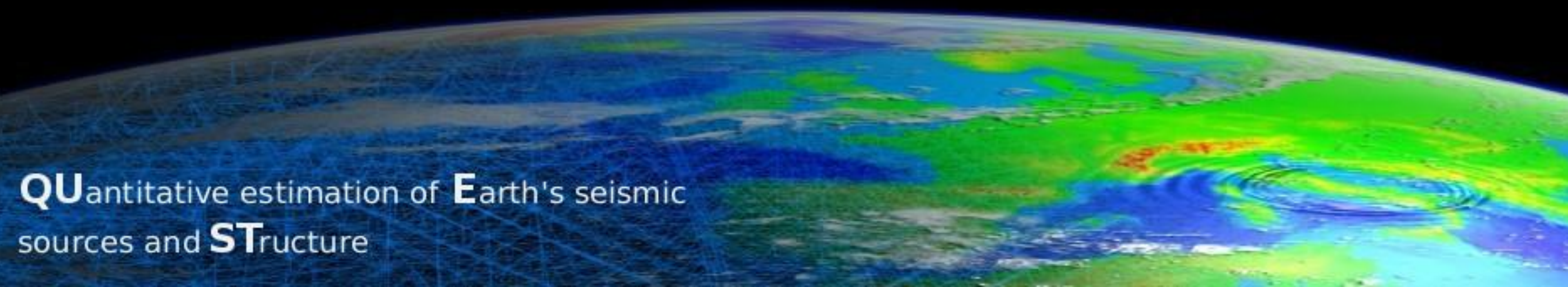
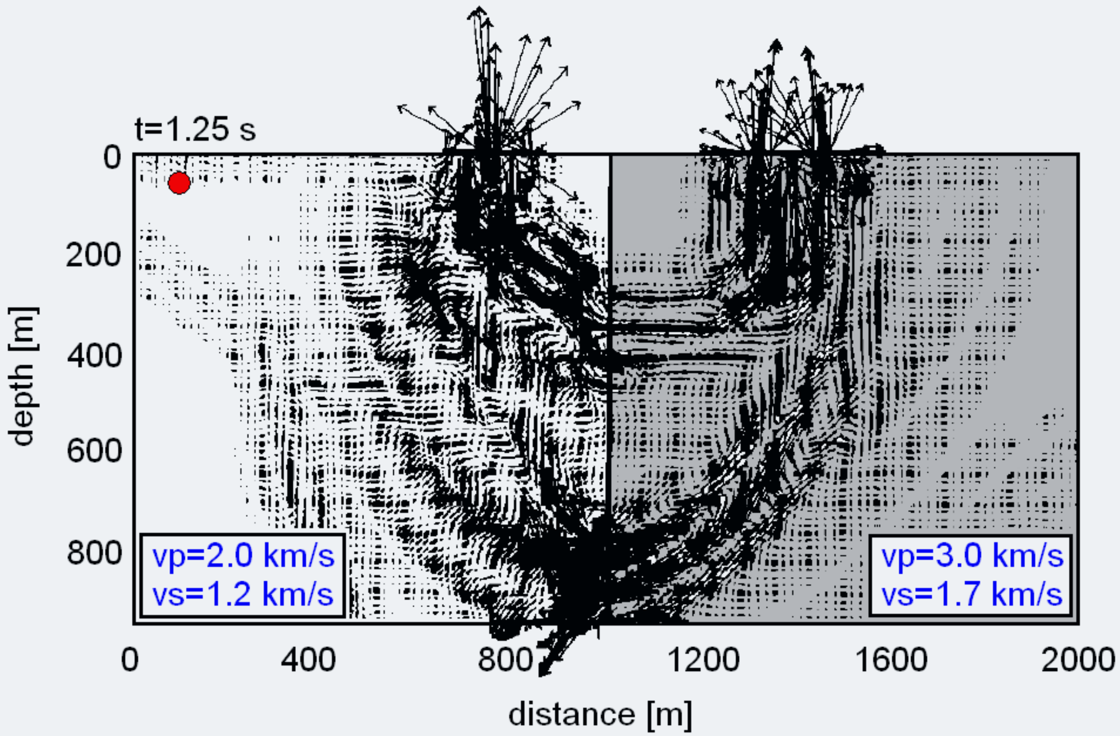


- **originally developed in fluid dynamics (Patera, 1984)**

- **migrated to seismology in the early 1990's (Seriani & Priolo, 1991)**

- **major advantage: accurate modelling of interfaces and the free surface (with topography)**

reflection and transmission of surfaces waves at material discontinuities




Priolo, Carcione & Seriani: 1994

## 1. The weak form of the wave equation …

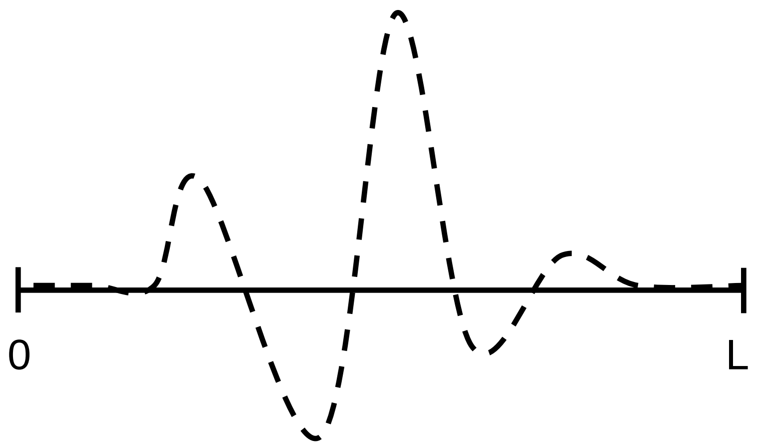
… in 1D

**SEM is based on the weak form of the wave equation**

$$\rho\ddot{u} - \frac{\partial}{\partial x}\left(\mu\frac{\partial}{\partial x}u\right) = f \qquad \frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,L) = 0$$

strong form of the wave equation

----------- PDE ----------- -- B.C. (free surface in 3D) --




vibrating string of length L

**SEM is based on the weak form of the wave equation**

$$\rho \ddot{u} - \frac{\partial}{\partial x}\left( \mu \frac{\partial}{\partial x} u \right) = f \qquad \frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,L) = 0$$

strong form of the wave equation

$$\int_0^L \rho\, w\, \ddot{u}\, dx - \int_0^L w \frac{\partial}{\partial x}\left( \mu \frac{\partial}{\partial x} u \right) dx = \int_0^L w\, f\, dx$$

multiply with test function w(x)

integrate over x from 0 to L

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multiply with test function w(x)

integrate over x from 0 to L

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_0^L w\, f\, dx$$

integrate by parts

use the boundary conditions

**SEM is based on the weak form of the wave equation**

**Solving the weak form of the wave equation means**

**to find a displacement field u(x,t) such that**

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_0^L w\, f\, dx$$

**is satisfied for any differentiable test function w(x).**

**Find a displacement field u(x,t) such that**

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_0^L w\, f\, dx$$

**is satisfied for any differentiable test function w(x).**

**Why is this important to know ?**

**Find a displacement field u(x,t) such that**

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_0^L w\, f\, dx$$

**is satisfied for any differentiable test function w(x).**

## 1. The basis of many numerical techniques:

- Finite-element method (FEM)
- Spectral-element method (SEM)
- Discontinuous Galerkin method (DGM)

**Find a displacement field u(x,t) such that**

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_0^L w\, f\, dx$$

**is satisfied for any differentiable test function w(x).**

**2. Free surface boundary condition is automatically satisfied**

**BIG advantage!**

## SEM in 1D: weak form of the wave equation

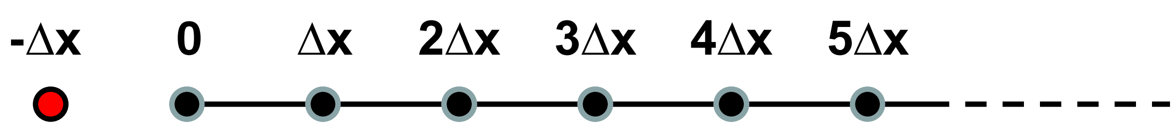
**Find a displacement field u(x,t) such that**

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left( \frac{\partial}{\partial x} w \right) \left( \frac{\partial}{\partial x} u \right) dx = \int_0^L w\, f\, dx$$

**is satisfied for any differentiable test function w(x).**

**2. Free surface boundary condition is automatically satisfied**

**BIG advantage!** Compare to finite-difference method:




$$0 = \frac{\partial}{\partial x} u(0)$$

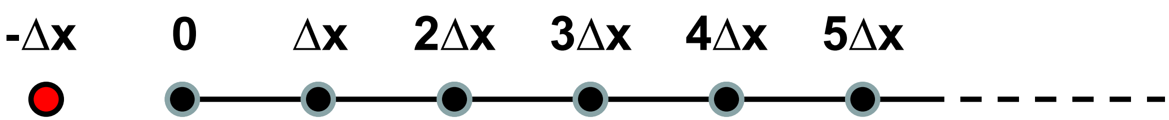
**Find a displacement field u(x,t) such that**

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_0^L w\, f\, dx$$

**is satisfied for any differentiable test function w(x).**

## 2. Free surface boundary condition is automatically satisfied

**BIG advantage!** Compare to finite-difference method:




$$0 = \frac{\partial}{\partial x} u(0) \approx \frac{u(\Delta x) - u(-\Delta x)}{2\Delta x}$$

Implementation of the free surface in FD requires grid points outside the computational domain!

**Find a displacement field u(x,t) such that**

$$\int_0^L \rho\, w\, \ddot{u}\, dx + \int_0^L \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_0^L w\, f\, dx$$

**is satisfied for any differentiable test function w(x).**

**2. Free surface boundary condition is automatically satisfied**

**BIG advantage!**

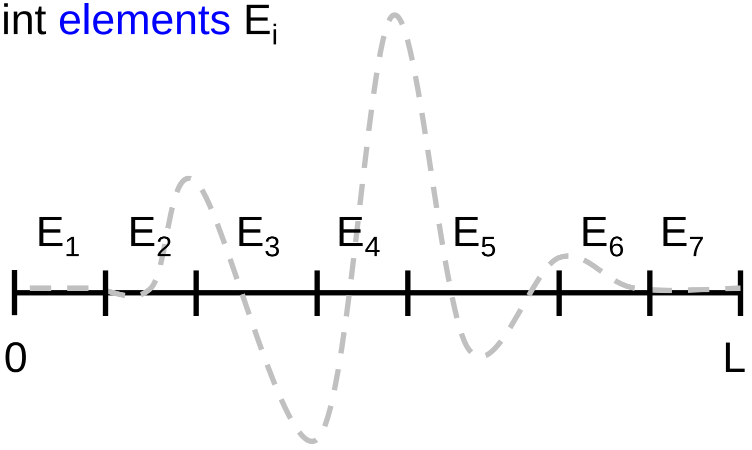
Correct free surface comes without any additional effort!

➡ This makes accurate surface waves!

## 2. Spatial discretisation

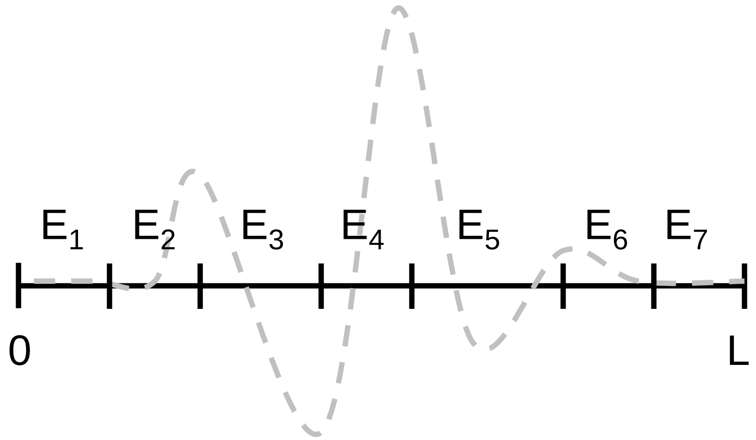
**1.** decompose the computational domain [0, L] into disjoint elements $E_i$

**2.** consider integral element-wise



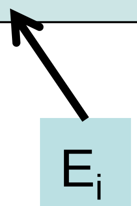

$$\int_{E_i} \rho\, w\, \ddot{u}\, dx + \int_{E_i} \mu \left( \frac{\partial}{\partial x} w \right) \left( \frac{\partial}{\partial x} u \right) dx = \int_{E_i} w\, f\, dx$$

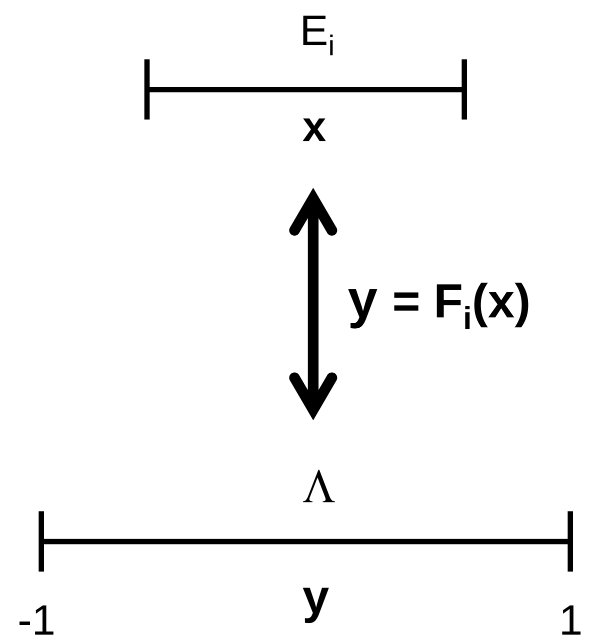
$E_i$




**3.** map each element to the reference interval [-1, 1]

$$\int_{E_i} \rho\, w\, \ddot{u}\, dx + \int_{E_i} \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_{E_i} w\, f\, dx$$



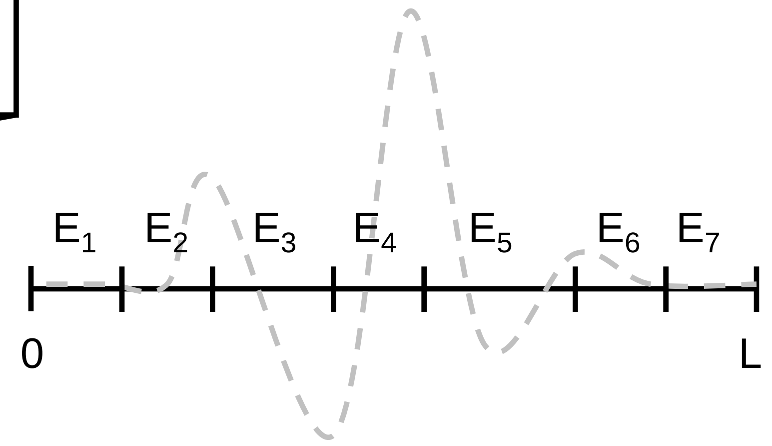
# SEM in 1D: mapping to the reference interval

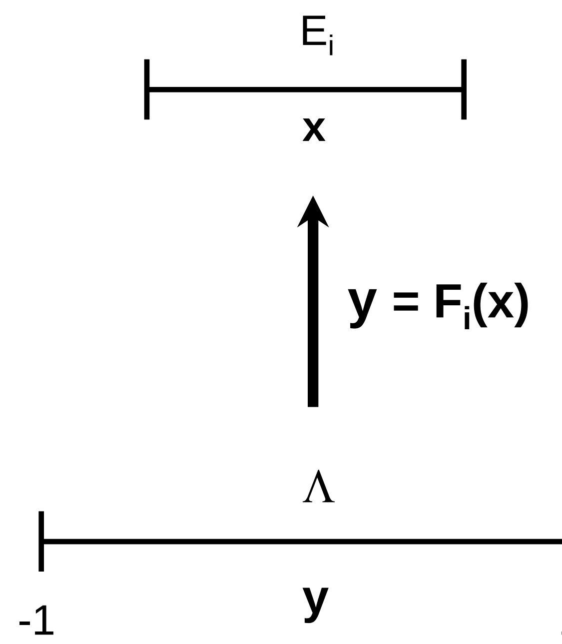
Is the same for every element !!!

All elements can now be treated in the same way.

$$\int_{-1}^{-1} \rho\, w\, \ddot{u} \frac{dx}{dy}\, dy + \dots$$

$$\int_{E_i} \rho\, w\, \ddot{u}\, dx + \int_{E_i} \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_{E_i} w\, f\, dx$$



**4.** Approximate u‘ by interpolating polynomials.

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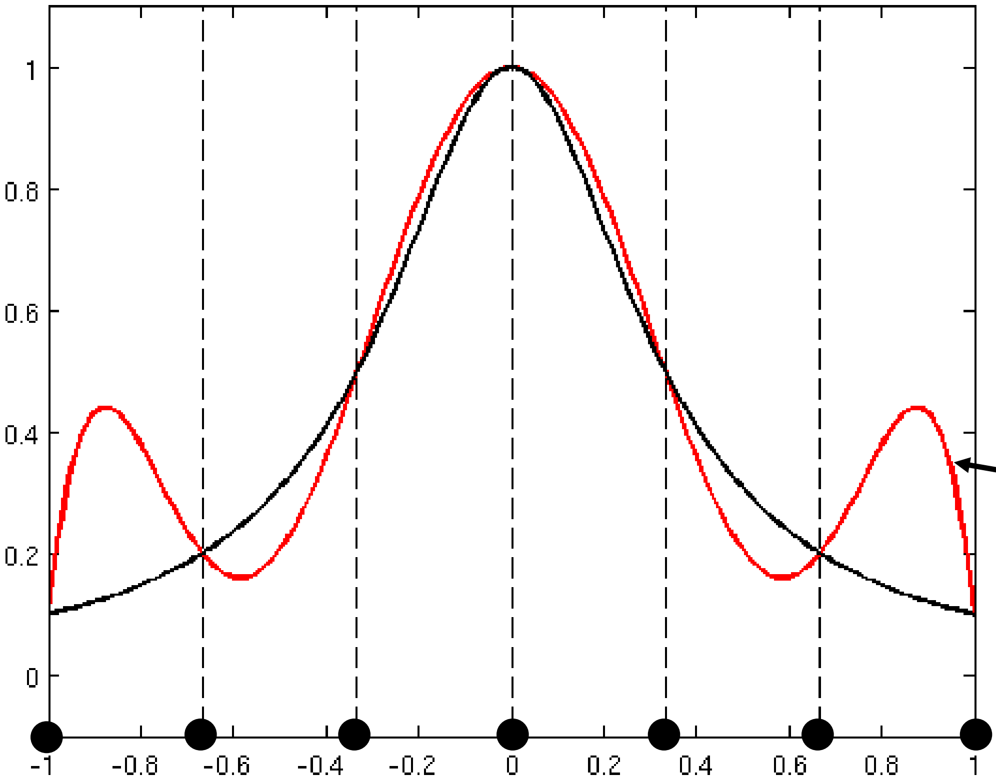



**<u>1st try:</u>** <u>equidistant collocation points</u>

exact wave field

approximation (interpolant)

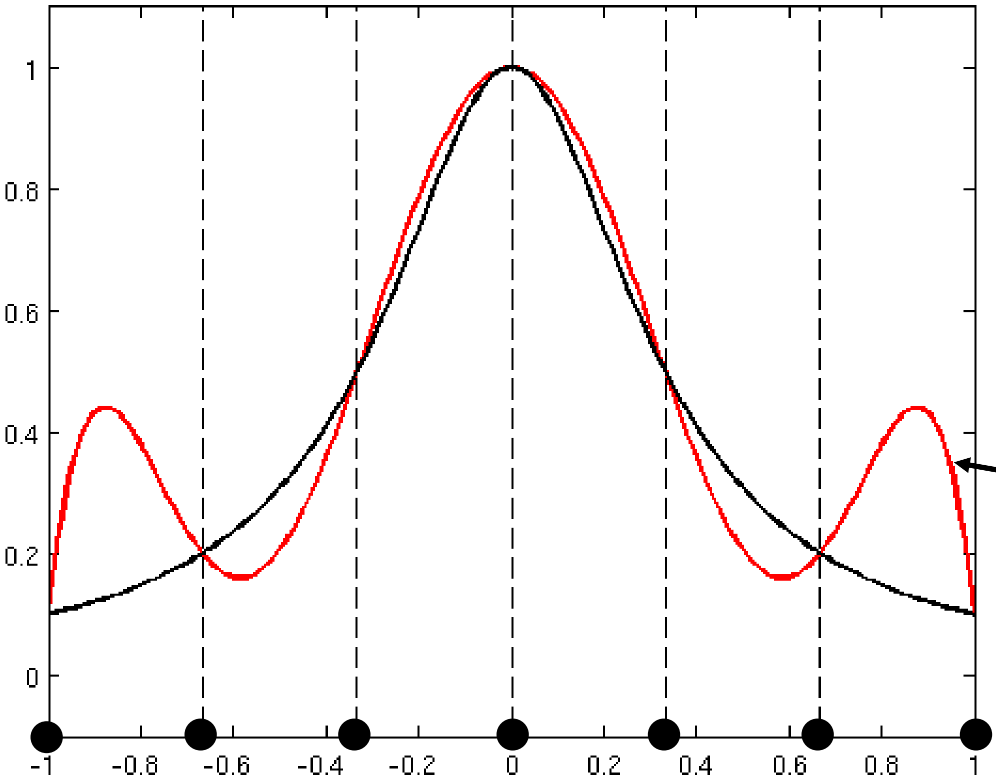
degree 6

7 collocation points

**Runge‘s phenomenon**

**4.** Approximate u' by interpolating polynomials.

$$\int_{-1}^{-1} \rho\, w\, \ddot{u} \frac{dx}{dy} dy + \dots$$




**<u>1st try: equidistant collocation points</u>**

exact wave field

approximation (interpolant)

degree 6

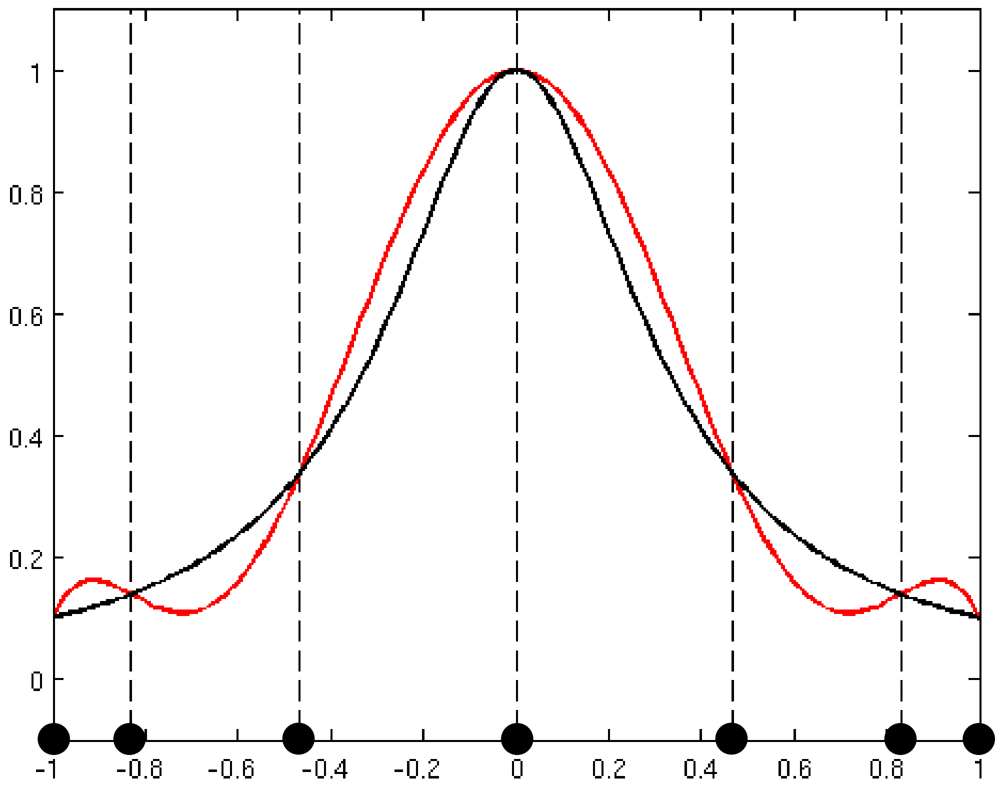
7 collocation points

**Runge's phenomenon**

**Don't use equidistant collocation points!**

**4.** Approximate u‘ by interpolating polynomials.

$$\int_{-1}^{-1} \rho\, w\, \ddot{u} \frac{dx}{dy} dy + \ldots$$




**2nd try:** Gauss-Lobatto-Legendre (GLL) points

exact wave field

approximation (interpolant)

degree 6

7 collocation points

**GLL collocation points avoid Runge‘s phenomenon!**

# SEM in 1D: approximation by polynomials

**4.** Approximate u by Lagrange polynomials collocated at the GLL points

**5.** choose Lagrange polynomials also for the test function w

$$\int_{-1}^{-1} \rho \, w \, \ddot{u} \, \frac{dx}{dy} \, dy + \ldots$$

$$\ell_k(y) \qquad u'(y,t) = \sum_{i=0}^{N} u_i(t) \, \ell_i(y)$$

**4.** Approximate u by Lagrange polynomials collocated at the GLL points

**5.** choose Lagrange polynomials also for the test function w

$$\int_{-1}^{-1} \rho\, w\, \ddot{u} \frac{dx}{dy} dy + \ldots$$

$$\ell_k(y) \quad u'(y,t) = \sum_{i=0}^{N} u_i(t)\, \ell_i(y)$$

$$\Rightarrow \quad \int_{-1}^{-1} \rho\, w\, \ddot{u} \frac{dx}{dy} dy = \sum_{i=0}^{N} \left[ \int_{-1}^{1} \rho\, \ell_k \ell_i \frac{dx}{dy} dy \right] \ddot{u}_i$$

$$\sum_{i=0}^{N} M_{ki}\, \ddot{u}_i(t)$$

**M=mass matrix**

# SEM in 1D: The mass matrix

**6.** The integral is approximated using Gauss-Lobatto-Legendre (GLL) quadrature.

Mass matrix is **diagonal** !!! **Big advantage !!!**

$$\int_{-1}^{-1} \rho\, w\, \ddot{u}\, \frac{dx}{dy}\, dy = \sum_{i=0}^{N} \left[ \int_{-1}^{1} \rho\, \ell_k \ell_i\, \frac{dx}{dy}\, dy \right] \ddot{u}_i$$

$$\sum_{i=0}^{N} M_{ki}\, \ddot{u}_i(t)$$

**M=mass matrix**

# SEM in 1D: A brief summary

$$\int_{E_i} \rho\, w\, \ddot{u}\, dx + \int_{E_i} \mu \left( \frac{\partial}{\partial x} w \right) \left( \frac{\partial}{\partial x} u \right) dx = \int_{E_i} w\, f\, dx$$

weak form, element-wise

1. mapping to the reference interval [-1 ,1]
2. polynomial approximation (GLL points)
3. numerical integration (GLL quadrature)

$$\sum_{i=0}^{N} M_{ki}\, \ddot{u}_i(t)$$

# SEM in 1D: A brief summary

$$\int_{E_i} \rho\, w\, \ddot{u}\, dx + \int_{E_i} \mu \left(\frac{\partial}{\partial x} w\right)\left(\frac{\partial}{\partial x} u\right) dx = \int_{E_i} w\, f\, dx$$

weak form, element-wise

repeat this for the remaining two terms …

$$\sum_{i=0}^{N} M_{ki}\, \ddot{u}_i(t) + \sum_{i=0}^{N} K_{ki}\, \ddot{u}_i = f_i$$

stiffness matrix

discrete force vector

# SEM in 1D: A brief summary

$$\int_{E_i} \rho\, w\, \ddot{u}\, dx - \int_{E_i} \mu \left( \frac{\partial}{\partial x} w \right) \left( \frac{\partial}{\partial x} u \right) dx = \int_{E_i} w\, f\, dx$$

weak form, element-wise

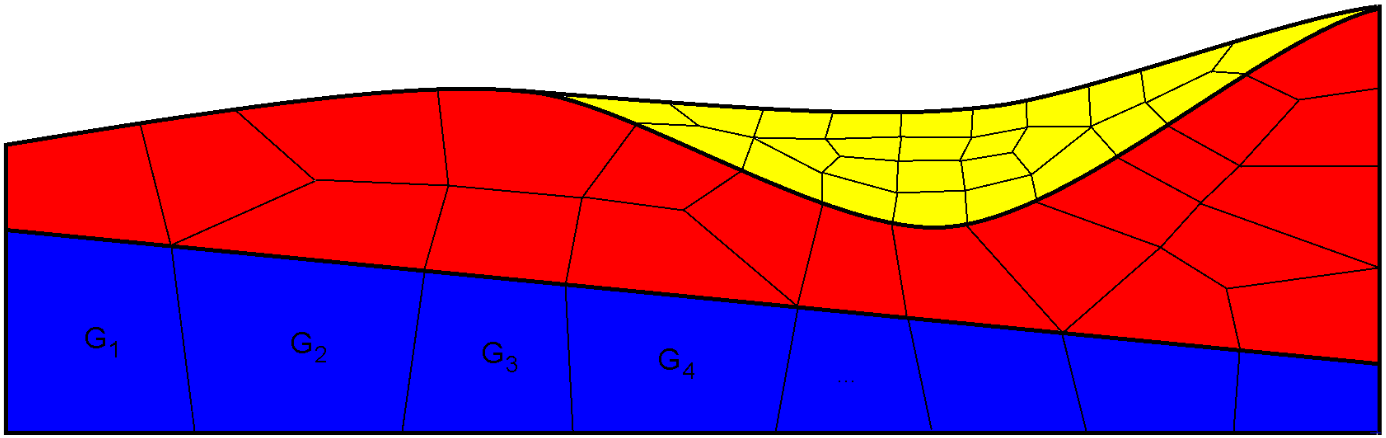
$$\ddot{\mathbf{u}} = \mathbf{M}^{-1} \cdot \left( \mathbf{f} - \mathbf{K} \cdot \mathbf{u} \right)$$

**partial differential equation** → spatial discretisation → **ordinary differential equation for the polynomial coefficients**

**You have survived the math part !**

## 3. The concept in 3D

**accurate solutions:** discontinuities need to coincide with element boundaries

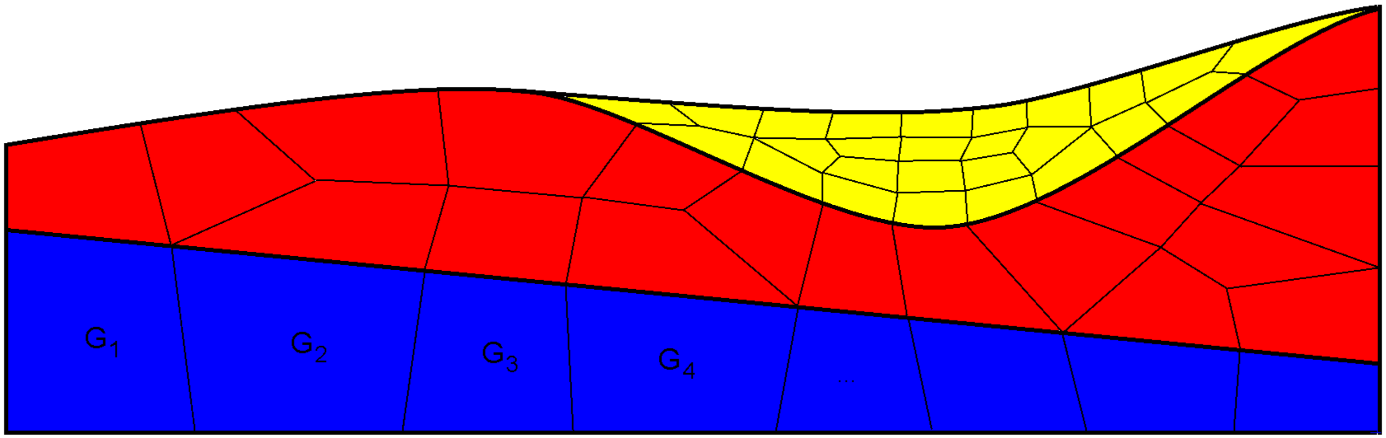



**low velocities:** short wavelength → small elements

**high velocities:** long wavelength → large elements

**many small elements → high computational costs !!!**

**accurate solutions:** discontinuities need to coincide with element boundaries



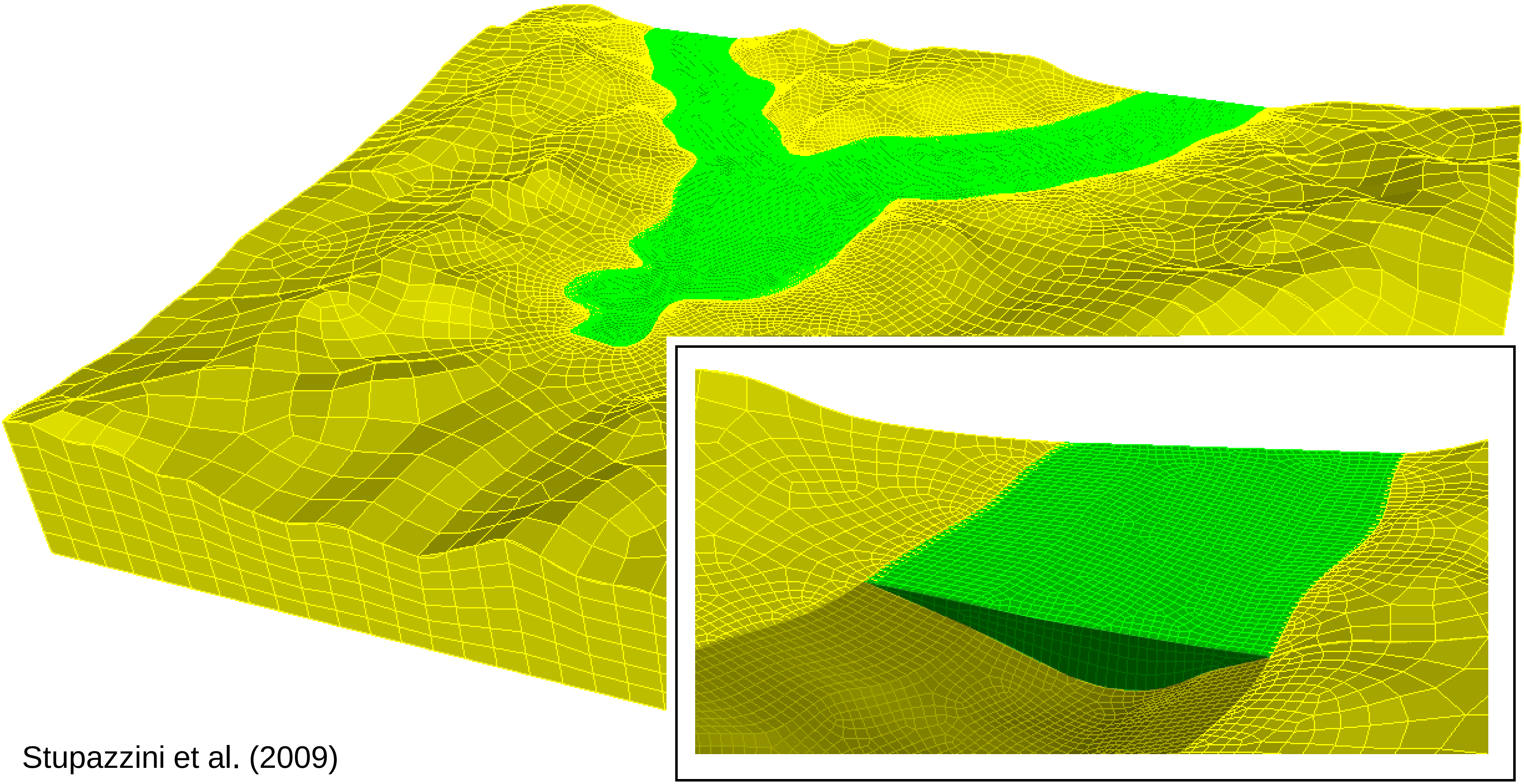

**low velocities:** short wavelength → small elements

**high velocities:** long wavelength → large elements

**many small elements → high computational costs !!!**

**possible solution: homogenisation theory (Y. Capdeville's talk)**

## Realistic example: The Grenoble valley



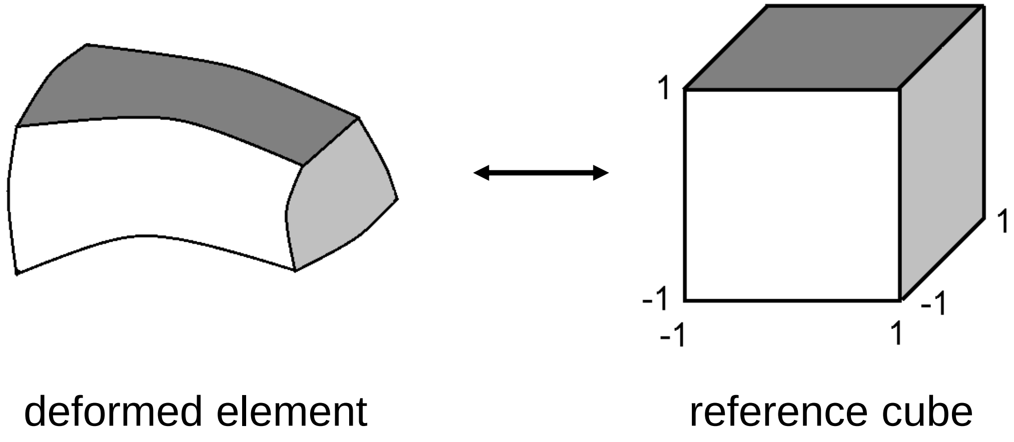
Stupazzini et al. (2009)

**Essentially the same as in 1D:**

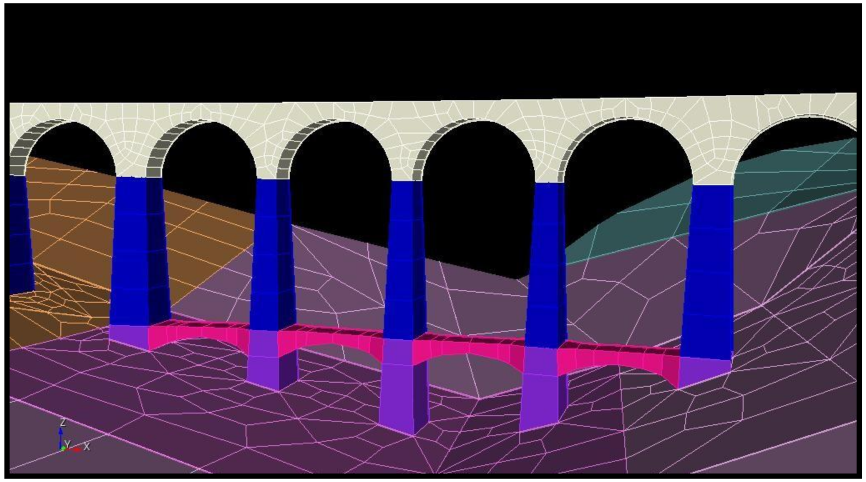
$$\rho \ddot{u}_i - \frac{\partial}{\partial x_j}\left( C_{ijkl} * \frac{\partial}{\partial x_k} u_l \right) = f_i$$

1. mapping to the reference cube $[-1\ ,1]^3$
2. polynomial approximation (GLL points)
3. numerical integration (GLL quadrature)

$$\ddot{\mathbf{u}} = \mathbf{M}^{-1} \cdot \left( \mathbf{f} - \mathbf{K} \cdot \mathbf{u} \right)$$

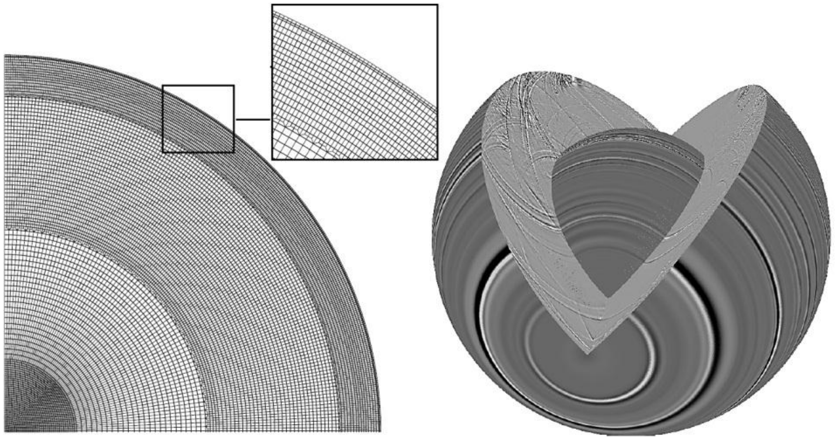


**GeoElse** (Milano, Cagliari)



- Interaction with engineering structures
- Nonlinear rheologies (visco-plasticity)

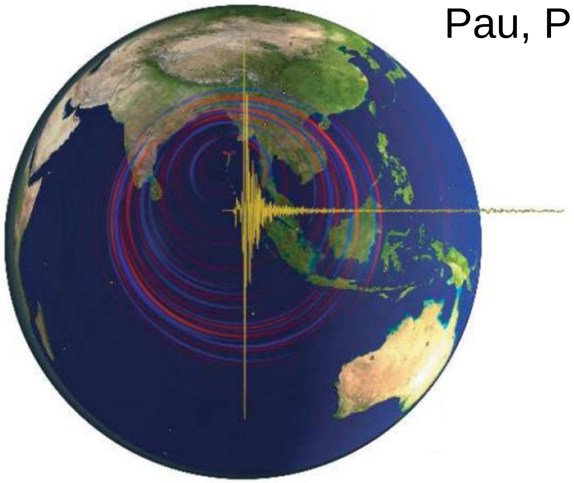
T. Nissen-Meyer: 2D domain, 3D synthetics



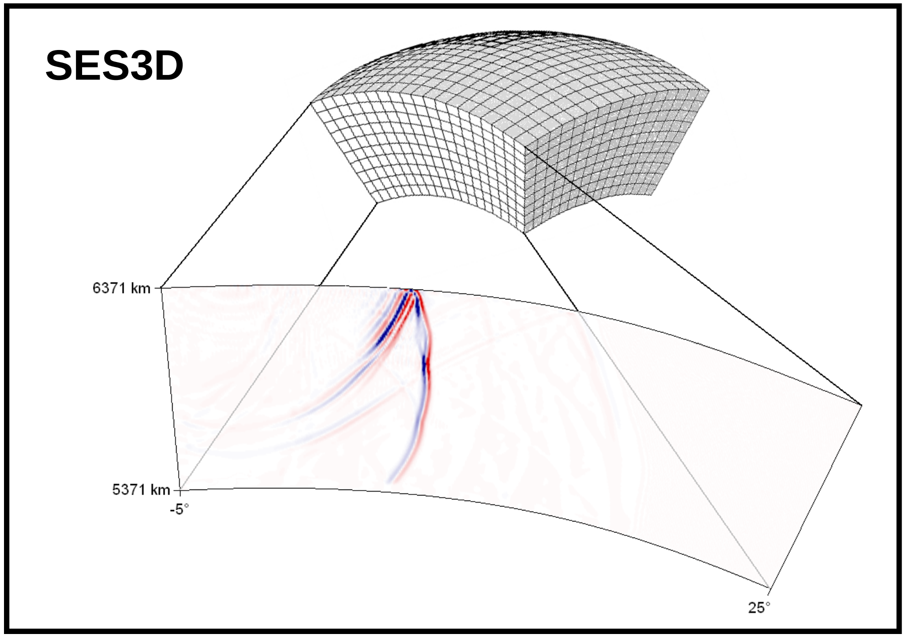
- 2D computational domain makes 3D synthetics
- Spherically symmetric Earth models
- High-frequency wave propagation

See the talk by J. Tromp and Q. Liu.

**SpecFEM3D** (Paris, Pau, Princeton)



- Spherical section, regular grid
- Simplistic and very easy to use
- Makes nice tomographic images



Thank you for your attention!