A poly-grid approach for wave propagation modeling in highly heterogeneous media by using a Chebyshev spectral element method

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Outline

Outline

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Modelling constraints

In exploration geophysics & engineering seismology

- heterogeneous properties must be correctly reproduced,
- very complex structures must be correctly modelled,
- numerical algorithms must be computationally efficient.

Spectral Element Method

Advantages

- high accuracy (spectral) & flexibility (finite element),
- a low value of G,
- highly accurate for very long propagation distances,
- numerical dispersion errors almost eliminated,
- fast solvers can be used.

Spectral Element Method

Advantages

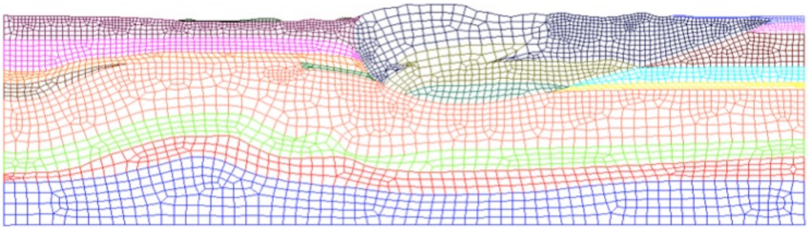
- high accuracy (spectral) & flexibility (finite element),
- a low value of G (number of grid points per minimum wavelength),
- highly accurate for very long propagation distances,
- numerical dispersion errors almost eliminated,
- fast solvers can be used.

Spectral Element Method

For wave modeling, a low value of G means:

- very coarse meshes are used,
- each single element may handle more than one of the shortest waves.

Spectral element mesh



- constant-property elements may reduce seriously the computational efficiency for subdomains with heterogeneous physical properties.

Outline

1. Introduction
   - Motivation
   - Poly-grid Spectral Element Method
2. Mathematical formulation
3. Examples
   - Acoustic
   - Elastic
4. Conclusion

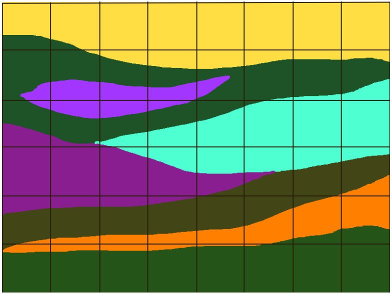
Poly-grid Spectral Element Method

Main properties:

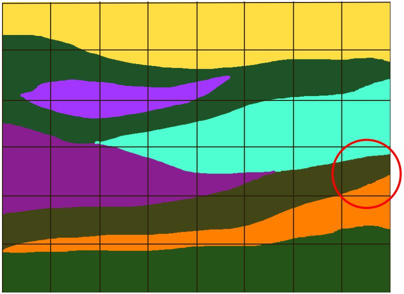
- large elements with high order (high accuracy),
- simple geometry of elements (easy discretization),
- property changes inside each element (high variability),
- changes that can be continuous or abrupt,
- changes that can be smaller than minimum wavelength.

Poly-grid Spectral Element Method

Poly-grid Spectral Element Method

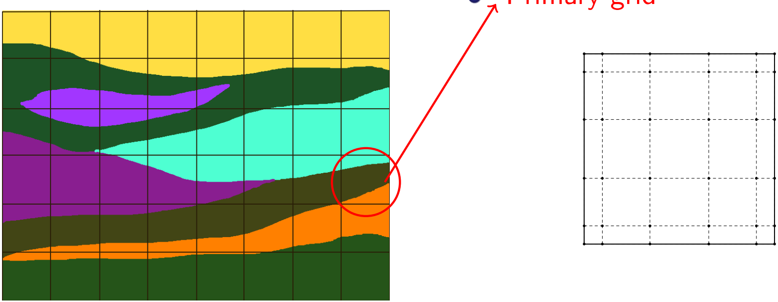


Poly-grid Spectral Element Method



Poly-grid Spectral Element Method

- Primary grid

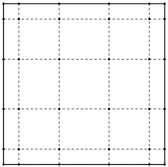


Poly-grid Spectral Element Method

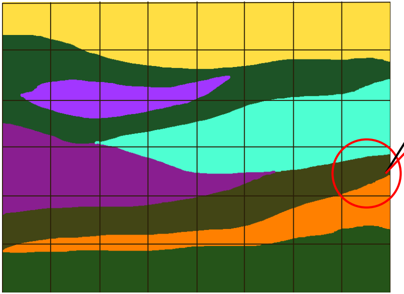
Primary grid

for computing the wave field:

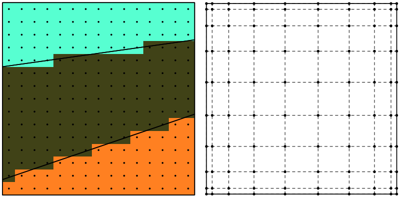
$$\tilde{u}^e(\xi_1,\xi_2,t) = \sum_{i_1=0}^{N}\sum_{i_2=0}^{N} \tilde{u}^e_{i_1 i_2}(t)\varphi_{i_1}(\xi_1)\varphi_{i_2}(\xi_2)$$



Poly-grid Spectral Element Method



- Primary grid
- Auxiliary (Extra) grids



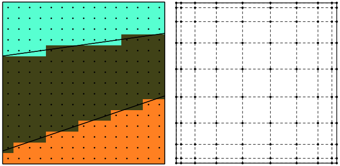
Poly-grid Spectral Element Method

Auxiliary grids

for describing physical parameters or force:

$$\tilde{\alpha}^e(\xi_1,\xi_2) = \sum_{l_1=0}^{L}\sum_{l_2=0}^{L} \tilde{\alpha}^e_{l_1 l_2}\phi_{l_1}(\xi_1)\phi_{l_2}(\xi_2),$$

$$\tilde{f}^e(\xi_1,\xi_2,t) = \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} f^e_{k_1 k_2}(t)\,\psi_{k_1}(\xi_1)\,\psi_{k_2}(\xi_2)$$



Outline

Wave Equation

Strong form

Acoustic

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho c^2}\frac{\partial u}{\partial t}\right) - \nabla \cdot \left(\frac{1}{\rho}\nabla u\right) = f$$

Elastic

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{D}^\top \mathbf{C}\mathbf{D}\mathbf{u} = \mathbf{f}$$

Wave Equation

Weak form

Acoustic

Find $u(\mathbf{x},t)$ solution of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(w,\frac{1}{\rho c^2}u)_\Omega + a(w,u)_\Omega = (w,f)_\Omega$$

with

$$(w,\frac{1}{\rho c^2}u)_\Omega = \int_\Omega w\frac{1}{\rho c^2}u\mathrm{d}\Omega,$$

$$a(w,u)_\Omega = \int_\Omega \nabla w\cdot(\frac{1}{\rho}\nabla u)\mathrm{d}\Omega,$$

$$(w,f)_\Omega = \int_\Omega wf\mathrm{d}\Omega.$$

Wave Equation

Weak form

Elastic

find the solution $\mathbf{u}(\mathbf{x},t)$ of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{w},\rho\mathbf{u})_\Omega + a(\mathbf{w},\mathbf{u})_\Omega = (\mathbf{w},\mathbf{f})_\Omega$$

with

$$(\mathbf{w},\rho\mathbf{u})_\Omega = \int_\Omega \rho\mathbf{w}^\top \cdot \mathbf{u}\ \mathrm{d}\Omega$$

$$a(\mathbf{w},\mathbf{u})_\Omega = \int_\Omega \varepsilon(\mathbf{w})^\top \sigma(\mathbf{u})\ \mathrm{d}\Omega = \int_\Omega \mathbf{w}^\top \mathbf{D}^\top \mathbf{C}\mathbf{D}\mathbf{u}\ \mathrm{d}\Omega$$

$$(\mathbf{w},\mathbf{f})_\Omega = \int_\Omega \mathbf{w}^\top \cdot \mathbf{f}\ \mathrm{d}\Omega$$

A system of Second-order Ordinary Differential Equations

Acoustic

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t)$$

Elastic

$$\begin{cases} \mathbf{M}\ddot{\mathbf{U}}_1(t) + \mathbf{K}_1\mathbf{U}_1(t) + \mathbf{K}_2\mathbf{U}_2(t) = \mathbf{F}_1(t) \\ \mathbf{M}\ddot{\mathbf{U}}_1(t) + \mathbf{K}_2^\top\mathbf{U}_1(t) + \mathbf{K}_3\mathbf{U}_2(t) = \mathbf{F}_2(t) \end{cases}$$

$$\mathbf{M} = \sum_{e=1}^{n_e} \mathbf{M}^e \qquad \mathbf{K} = \sum_{e=1}^{n_e} \mathbf{K}^e \qquad \mathbf{F} = \sum_{e=1}^{n_e} \mathbf{F}^e$$

mass matrix   stiffness matrix   force vector

Element Matrices

- Acoustic

$$M^e_{i_1 i_2 j_1 j_2} = \frac{\Delta_1^e}{2}\frac{\Delta_2^e}{2}\sum_{l_1=0}^{L_1}\sum_{l_2=0}^{L_2}\tilde{\alpha}^e_{l_1 l_2} B_{i_1 j_1 l_1} B_{i_2 j_2 l_2}, \quad \text{with } \alpha = \frac{1}{\rho c^2}$$

$$K^e_{i_1 i_2 j_1 j_2} = \sum_{l_1=0}^{L_1}\sum_{l_2=0}^{L_2}\tilde{\beta}^e_{l_1 l_2}\left(\frac{\Delta_2^e}{\Delta_1^e}\bar{B}_{i_1 j_1 l_1} B_{i_2 j_2 l_2} + \frac{\Delta_1^e}{\Delta_2^e} B_{i_1 j_1 l_1}\bar{B}_{i_2 j_2 l_2}\right)$$

$$\text{with } \beta = \frac{1}{\rho}$$

$$F^e_{i_1 i_2} = \frac{\Delta_1^e}{2}\frac{\Delta_1^e}{2}\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2} A_{i_1 k_1} A_{i_2 k_2}\tilde{f}^e_{k_1 k_2}(t)$$

Element Matrices

- Elastic

$$M^e_{i_1 i_2 j_1 j_2} = \frac{\Delta^e_1 \Delta^e_2}{4} \sum_{l_1=0}^{L_1} \sum_{l_2=0}^{L_2} \tilde{\rho}^e_{l_1 l_2} B_{i_1 j_1 l_1} B_{i_2 j_2 l_2}$$

$$[\mathbf{K}^e_1]_{i_1 i_2 j_1 j_2} = \sum_{l_1=0}^{L_1} \sum_{l_2=0}^{L_2} \left[(\lambda_{l_1 l_2} + 2\mu_{l_1 l_2}) \frac{\Delta^e_2}{\Delta^e_1} \bar{\bar{B}}_{i_1 j_1 l_1} B_{i_2 j_2 l_2} \; + \; \mu_{l_1 l_2} \frac{\Delta^e_1}{\Delta^e_2} B_{i_1 j_1 l_1} \bar{\bar{B}}_{i_2 j_2 l_2} \right]$$

$$[\mathbf{K}^e_2]_{i_1 i_2 j_1 j_2} = \sum_{l_1=0}^{L1} \sum_{l_2=0}^{L_2} \left(\lambda_{l_1 l_2} \bar{B}_{i_1 j_1 l_1} \bar{B}_{j_2 i_2 l_2} \; + \; \mu_{l_1 l_2} \bar{B}_{j_1 i_1 l_1} \bar{B}_{i_2 j_2 l_2} \right)$$

$$[\mathbf{K}^e_3]_{i_1 i_2 j_1 j_2} = \sum_{l_1=0}^{L_1} \sum_{l_2=0}^{L_2} \left[(\lambda_{l_1 l_2} + 2\mu_{l_1 l_2}) \frac{\Delta^e_1}{\Delta^e_2} B_{i_1 j_1 l_1} \bar{\bar{B}}_{i_2 j_2 l_2} \; + \; \mu_{l_1 l_2} \frac{\Delta^e_2}{\Delta^e_1} \bar{\bar{B}}_{i_1 j_1 l_1} B_{i_2 j_2 l_2} \right]$$

$$F^e_{i_1 i_2} = \frac{\Delta^e_1 \Delta^e_2}{4} \sum_{k_1=0}^{K_1} \sum_{k_2=0}^{K_2} A_{i_1 k_1} A_{i_2 k_2} f^e_{k_1 k_2}(t)$$

Poly-grid coupling operators

$$A_{ik} = \int_{-1}^{+1} \varphi_i(\xi)\, \psi_k(\xi)\, \mathrm{d}\xi$$

$$B_{ijl} = \int_{-1}^{+1} \phi_l(\xi)\, \varphi_i(\xi)\, \varphi_j(\xi)\, \mathrm{d}\xi$$

$$\bar{B}_{ijl} = \int_{-1}^{+1} \phi_l(\xi)\, \varphi_i(\xi)\, \frac{\mathrm{d}\varphi_j(\xi)}{\mathrm{d}\xi}\, \mathrm{d}\xi$$

$$\bar{\bar{B}}_{ijl} = \int_{-1}^{+1} \phi_l(\xi)\, \frac{\mathrm{d}\varphi_i(\xi)}{\mathrm{d}\xi}\, \frac{\mathrm{d}\varphi_j(\xi)}{\mathrm{d}\xi}\, \mathrm{d}\xi$$

Outline

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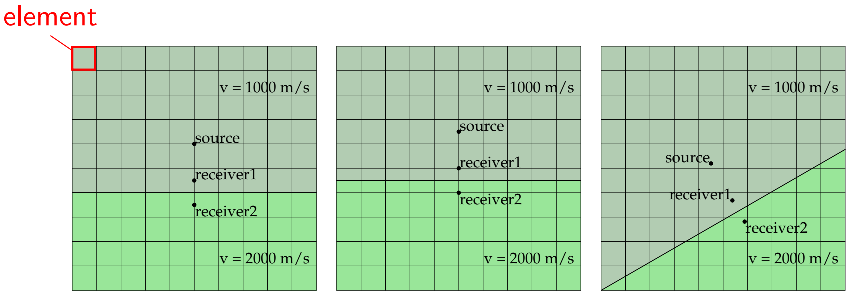
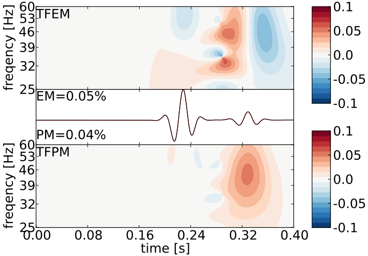
Acoustic wave impinging on a planar interface



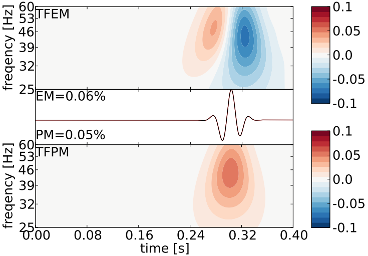
(a) model 1 (b) model 2 (c) model 3

Figure: Two media models with an interface in different positions with respect to the spatial discretization.

Comparing model2(PG-SEM) with model1(SEM)



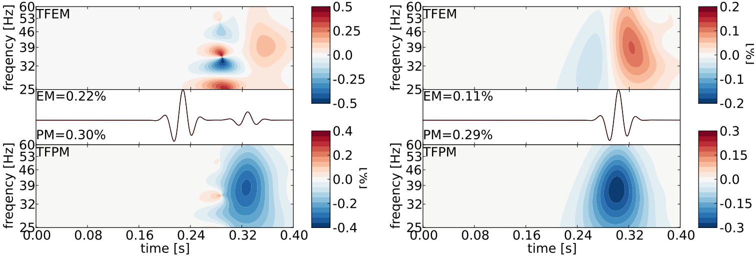
(a) recevier1



(b) receiver2

Figure: pressure collected at receivers, Time-frequency Misfits

Comparing model3(inclined interface, PG-SEM) with model1(SEM)

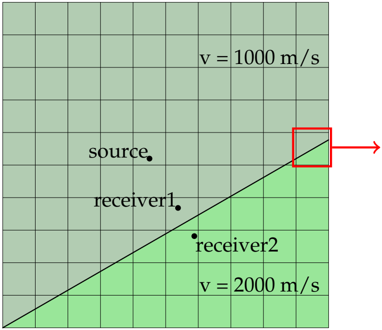


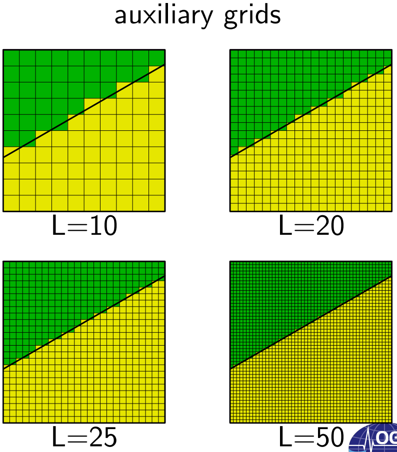
(a) recevier1

(b) receiver2

Figure: pressure collected at receivers, Time-frequency Misfits

Convergence with the use of increasing order to discretize the media





Convergence with the use of increasing order to discretize the media

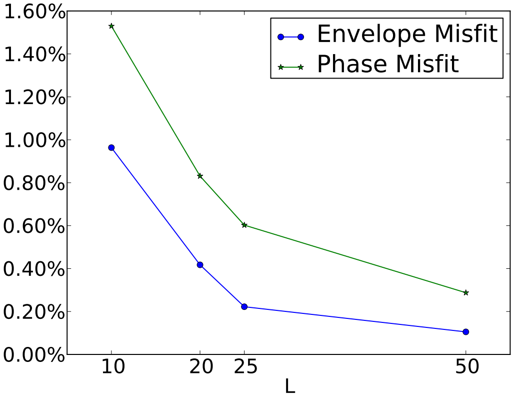


Figure: Envelope misfit and Phase misfit comparing to model1(SEM)

Plane wave impinging on a circular interface

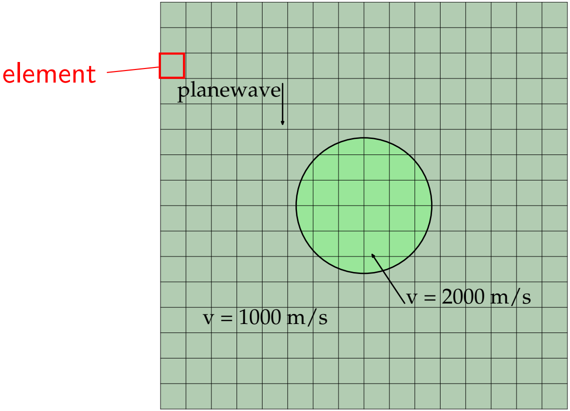
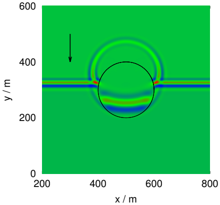
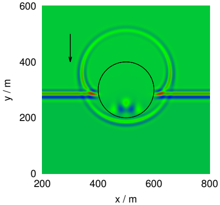


Figure: Model description

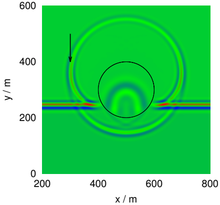
Plane wave impinging on a circular interface



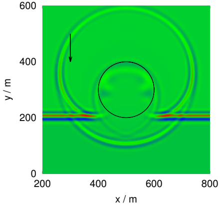
(a) $t = 0.220\text{s}$



(b) $t = 0.260\text{s}$

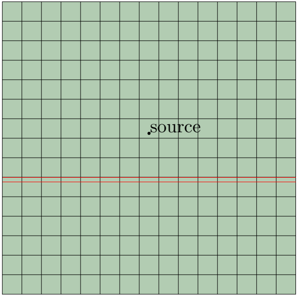


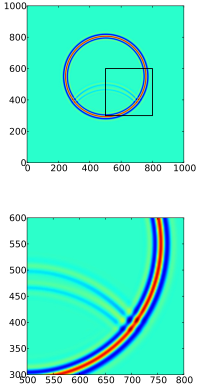
(c) $t = 0.300s$



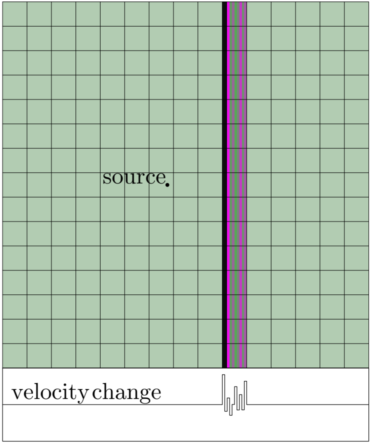
(d) $t = 0.340s$

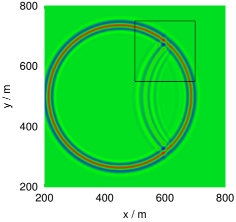
Wave propagating across thin layers

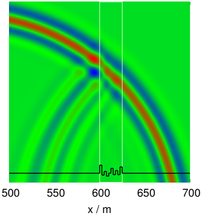




Wave propagating across thin layers







Plane wave diffraction by a wedge

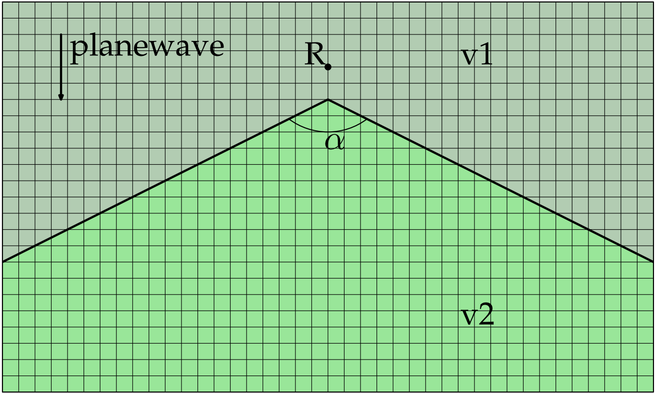


Figure: Description of the wedge model

Plane wave diffraction by a wedge

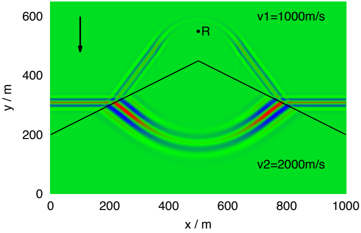


Figure: snapshot at $t = 0.350\mathrm{s}$

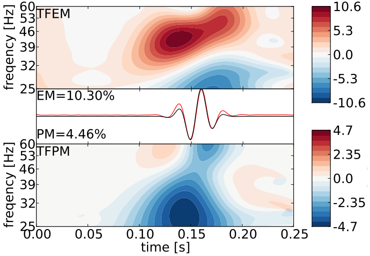


Figure: comparing the diffracted wave with analytical solution

Plane wave diffraction by a wedge

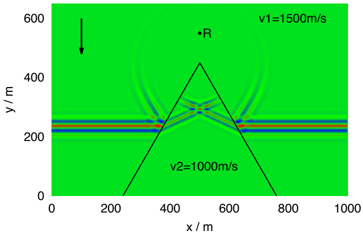
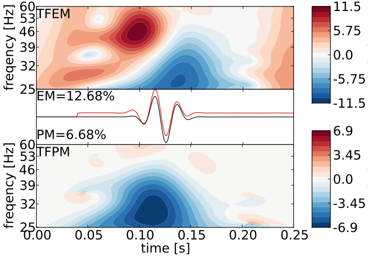


Figure: snapshot at $t = 0.300$s



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Plane wave diffraction by a wedge

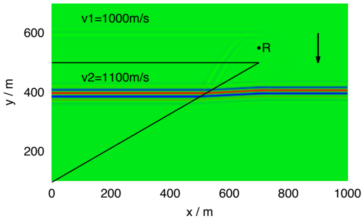


Figure: snapshot at $t = 0.250$s

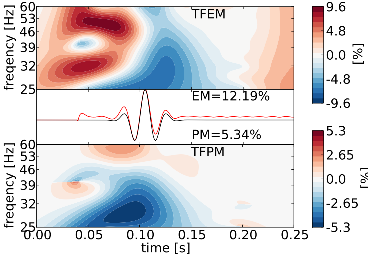


Figure: comparing the diffracted wave with analytical solution

Outline

Elastic wave impinging on a planar interface

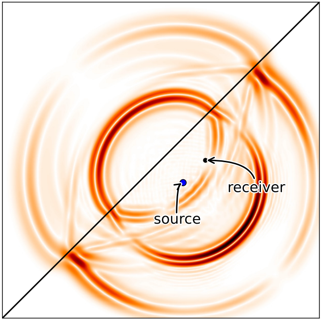


Figure: snapshot of particle velocity modulus

Elastic wave impinging on a planar interface

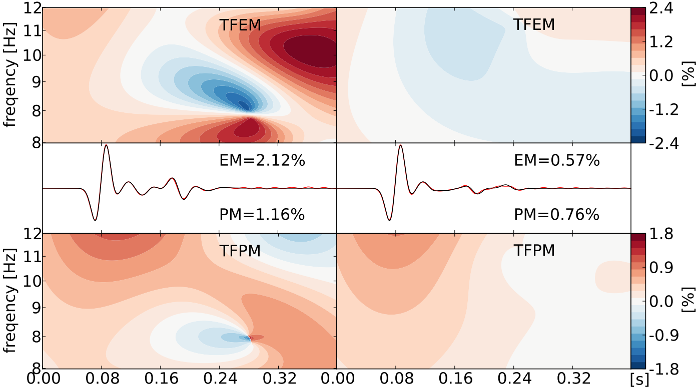


Figure: comparing the particle velocity with FDTD result

Elastic wave diffraction by a wedge

snapshots of particle velocity

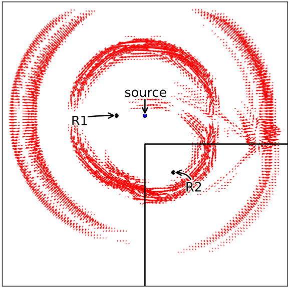


Figure: SEM

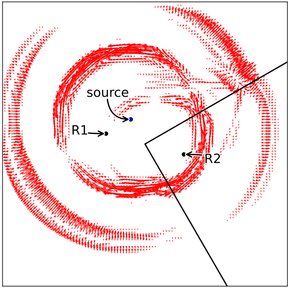
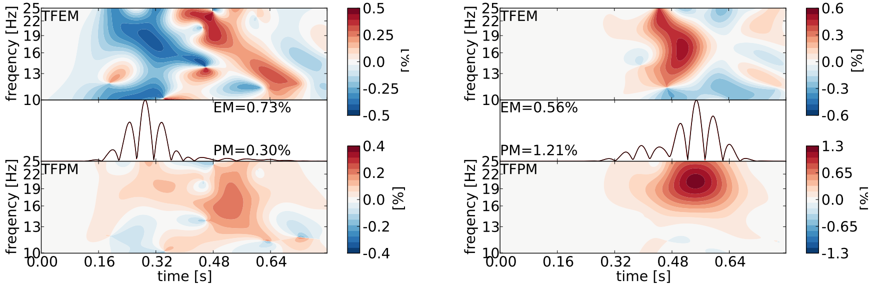


Figure: Poly-grid SEM

Elastic wave diffraction by a wedge

Comparing modulus of particle velocity between SEM and PG-SEM



Outline

Conclusion

Poly-grid Spectral Element Method is a computationally efficient and accurate scheme that allows for:

- solving problems characterized by small scale heterogeneous properties,
- much coarser grid computations than in other approaches,
- facilitated model preparation & modification.

Thanks for your attention.