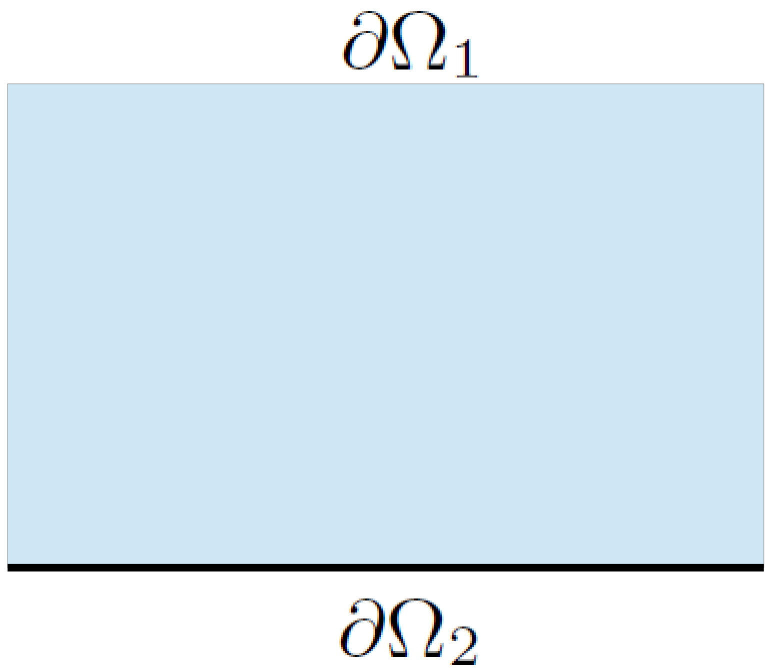
Higher-order adjoint methods and Backus-Gilbert Theory

David Al-Attar, University of Cambridge

A toy inverse problem



$$\rho\partial_t^2 u - \nabla\cdot\mathbf{t} = 0, \quad \mathbf{x}\in\Omega$$

$$\mathbf{t} = \mu\nabla u$$

$$\hat{\mathbf{n}}\cdot\mathbf{t} = 0, \quad \mathbf{x}\in\partial\Omega_1$$

$$u = s, \quad \mathbf{x}\in\partial\Omega_2$$

$$u(\mathbf{x},0) = \partial_t u(\mathbf{x},0)$$

$$A : L^2(\partial\Omega_2\times[0,T];\mathbb{R}) \to L^2(\partial\Omega_1\times[0,T];\mathbb{R})$$

$$s \mapsto A(s) = u|_{\partial\Omega_1\times[0,T]}$$

Optimization formulation

Objective functional:

$$J(u) = \frac{1}{2} \int_0^T \int_{\partial\Omega_1} w(u - u_{\mathrm{obs}})^2 \, \mathrm{d}S \, \mathrm{d}t$$

Reduced objective functional:

$$\hat{J}(s) = (J \circ A)(s)$$

Best-fitting model:

$$\tilde{s} = \arg\min_s \hat{J}(s)$$

Solution using adjoint methods

First-order necessary optimality condition:

$$D_s \hat{J}(\tilde{s}) = 0$$

Gradient calculated using first-order adjoint method

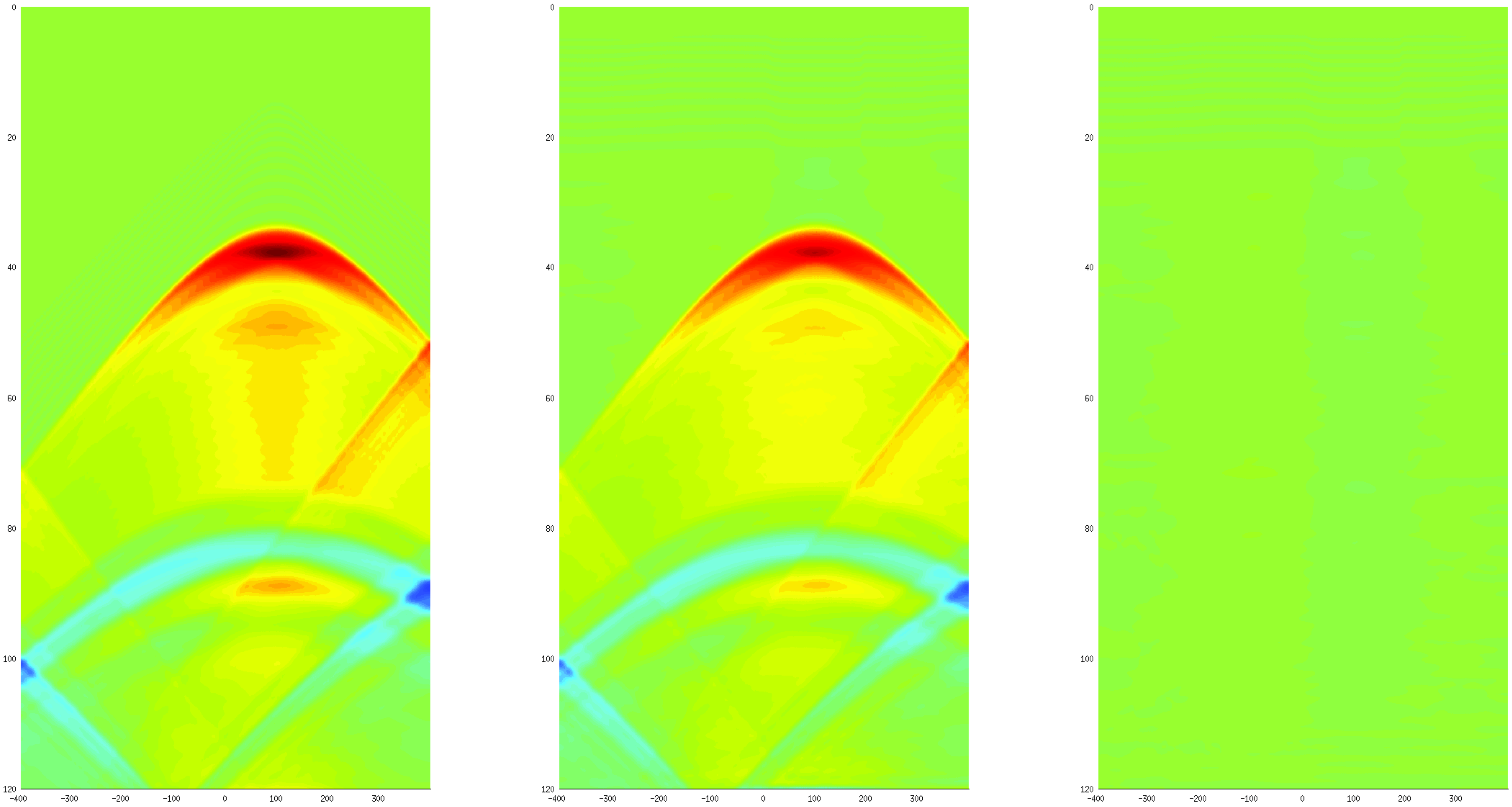
Second-order sufficient optimality condition:

$$D_s^2 \hat{J}(\tilde{s}) > 0$$

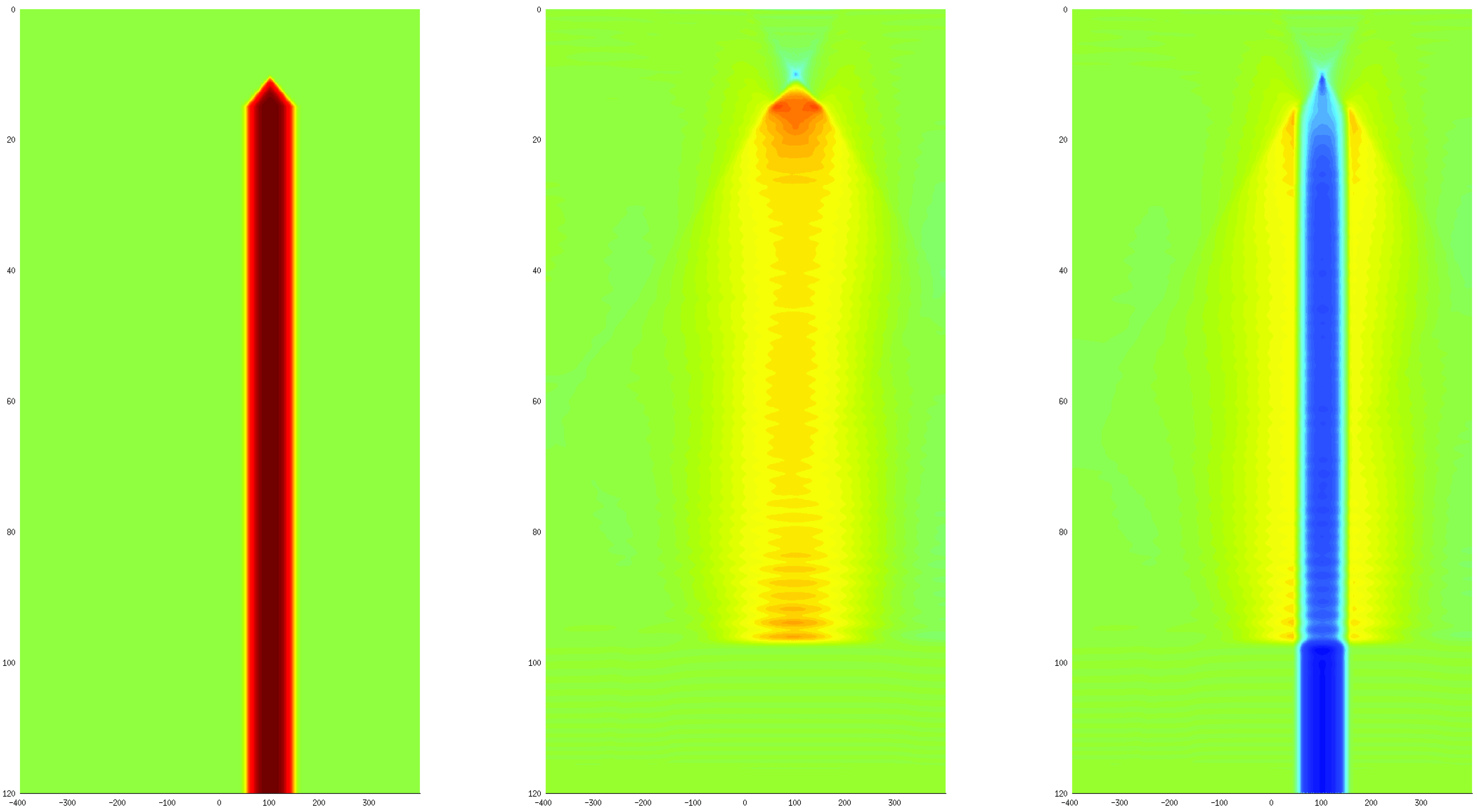
Hessian products calculated using second-order adjoint method

Iterative solution using gradient based methods

Data comparison



Slip comparison



Abstract form of problem

For $A \in L(X;Y)$ we consider:

$$Ax = y \qquad J(x) = \frac{1}{2}\|Ax - y\|_Y^2$$

Orthogonal decompositions: $\langle Ax, y\rangle_Y = \langle x, A^* y\rangle_X$

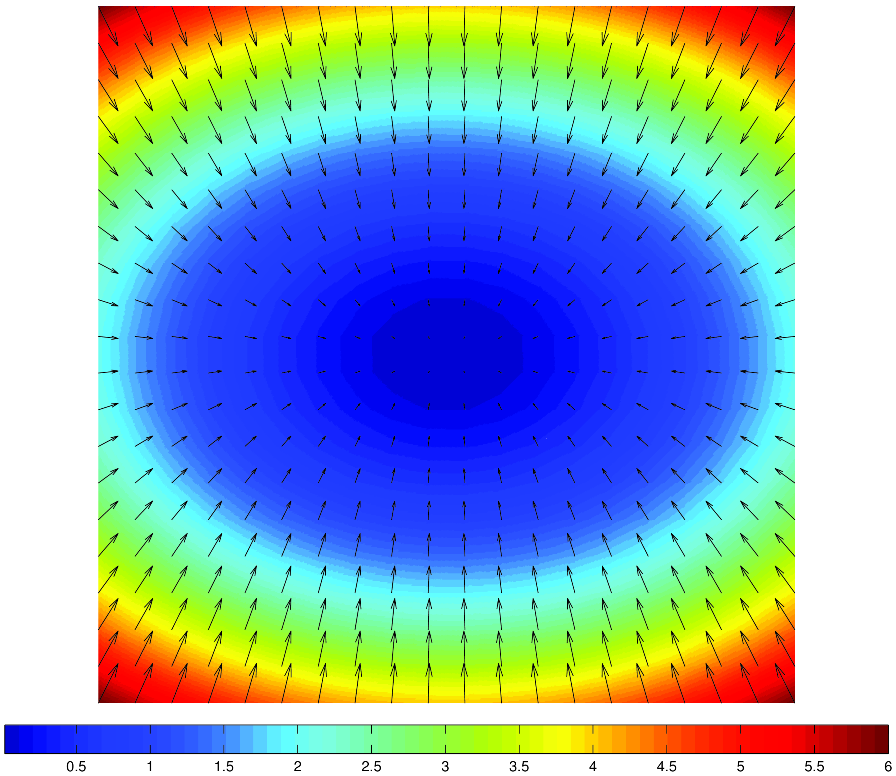
$$X = \ker A \oplus \mathrm{im} A^* \quad Y = \mathrm{im} A \oplus \ker A^*$$

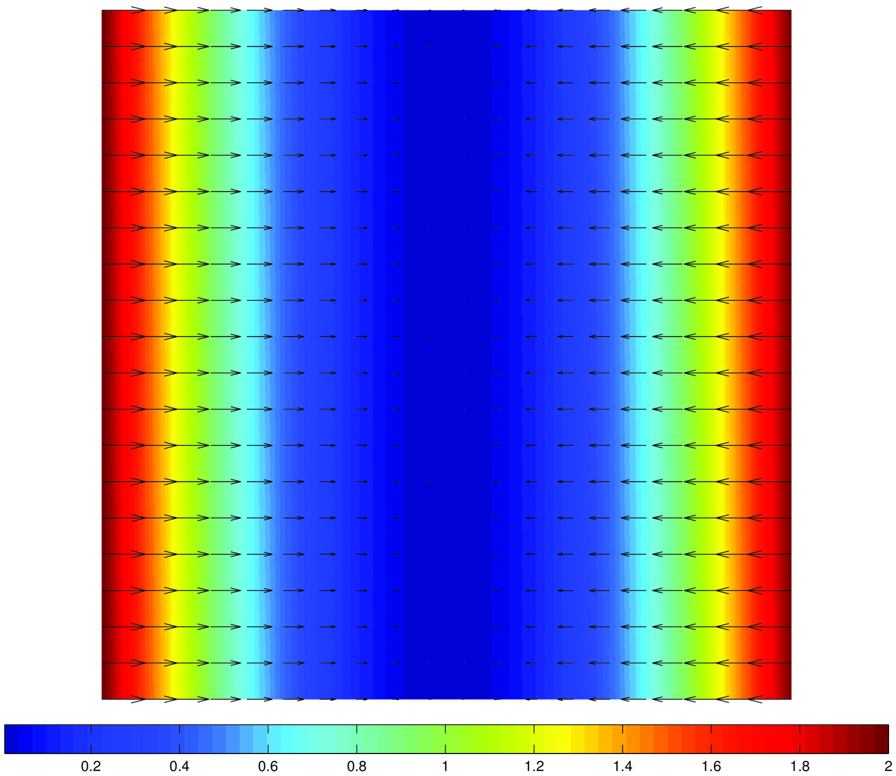
$$J(x) = \frac{1}{2}\|Ax - Py\|_Y^2 + \frac{1}{2}\|(1-P)y\|_Y^2$$

Gradient and Hessian:

$$DJ(x) = A^*(Ax - y) \quad D^2 J(x) = A^* A$$

Singular Hessians





Backus-Gilbert theory

From $Ax = y$ we have for $y' \in Y$

$$\langle y, y' \rangle_Y = \langle x, A^* y' \rangle_X$$

If $x' \in \mathrm{im} A^*$ we can compute $\langle x, x' \rangle_X$ from the data

For $x' \in X$ we can approximate $\langle x, x' \rangle_X$ by finding:

$$\tilde{y}' = \arg\min_{y' \in Y} \|A^* y' - x'\|_X^2$$

$$|\langle x, x' \rangle_X - \langle y, \tilde{y}' \rangle_Y| \leq \|A^* \tilde{y}' - x'\|_X \|x\|_X$$

Back to the toy problem

Linear objective functional:

$$J(u) = \int_0^T \int_{\partial\Omega_1} w u \, \mathrm{d}S \, \mathrm{d}t$$

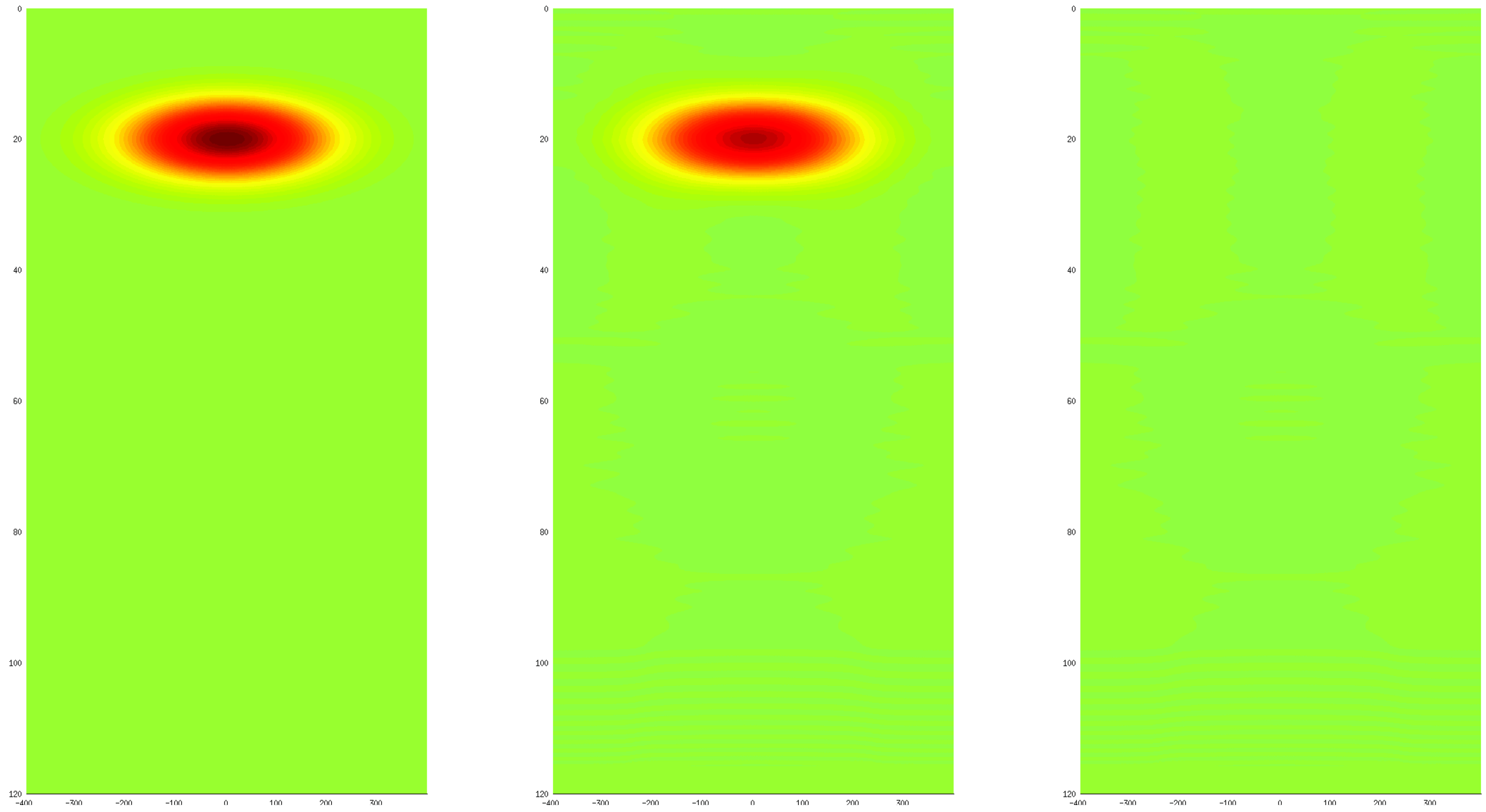
Derivative of reduced objective functional:

$$D_s \hat{J}(s)\{\delta s\} = \int_0^T \int_{\partial\Omega_2} K_s \delta s \, \mathrm{d}S \, \mathrm{d}t$$

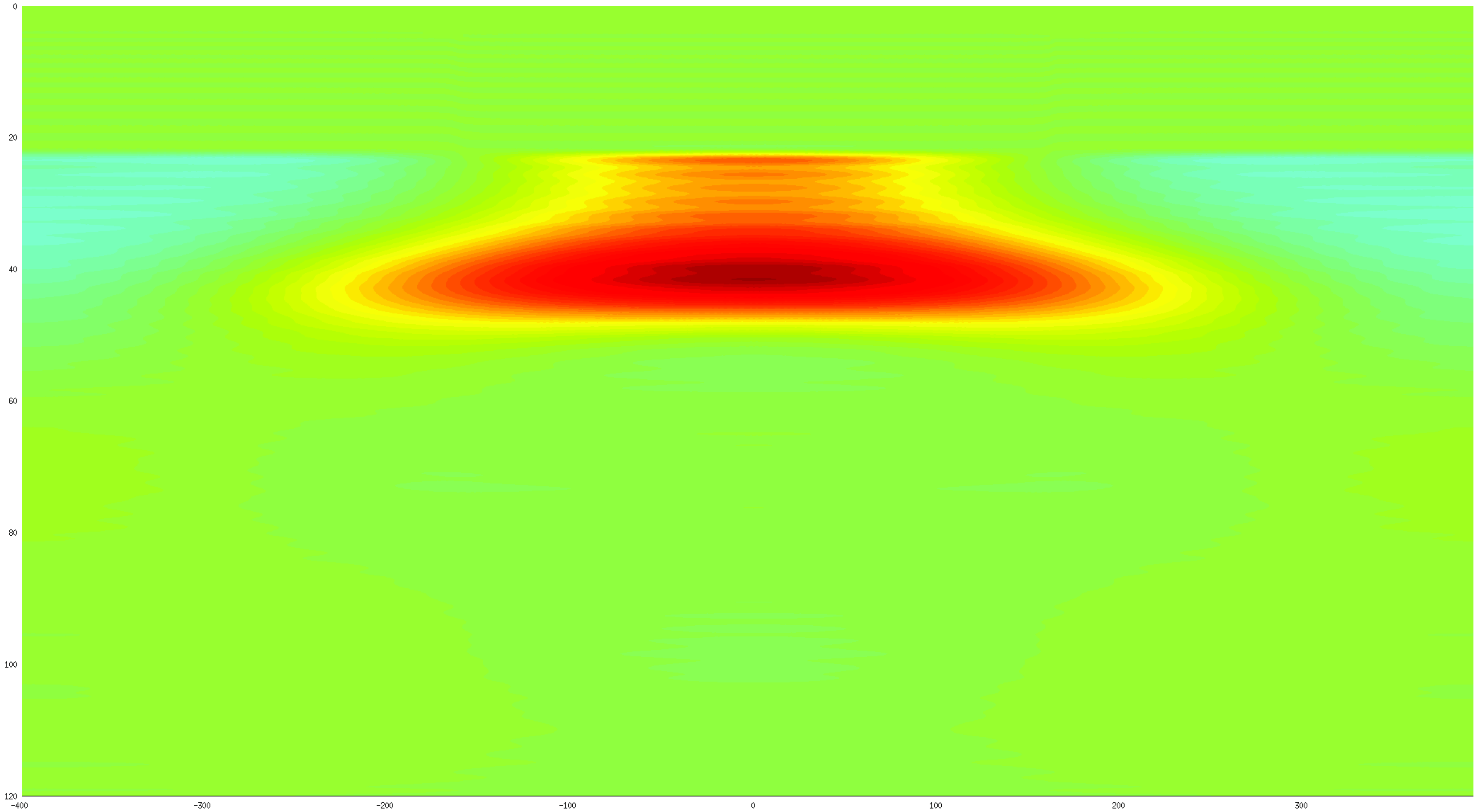
Secondary objective functional

$$J'(K_s) = \frac{1}{2} \int_0^T \int_{\partial\Omega_2} (K_s - h)^2 \, \mathrm{d}S \, \mathrm{d}t$$

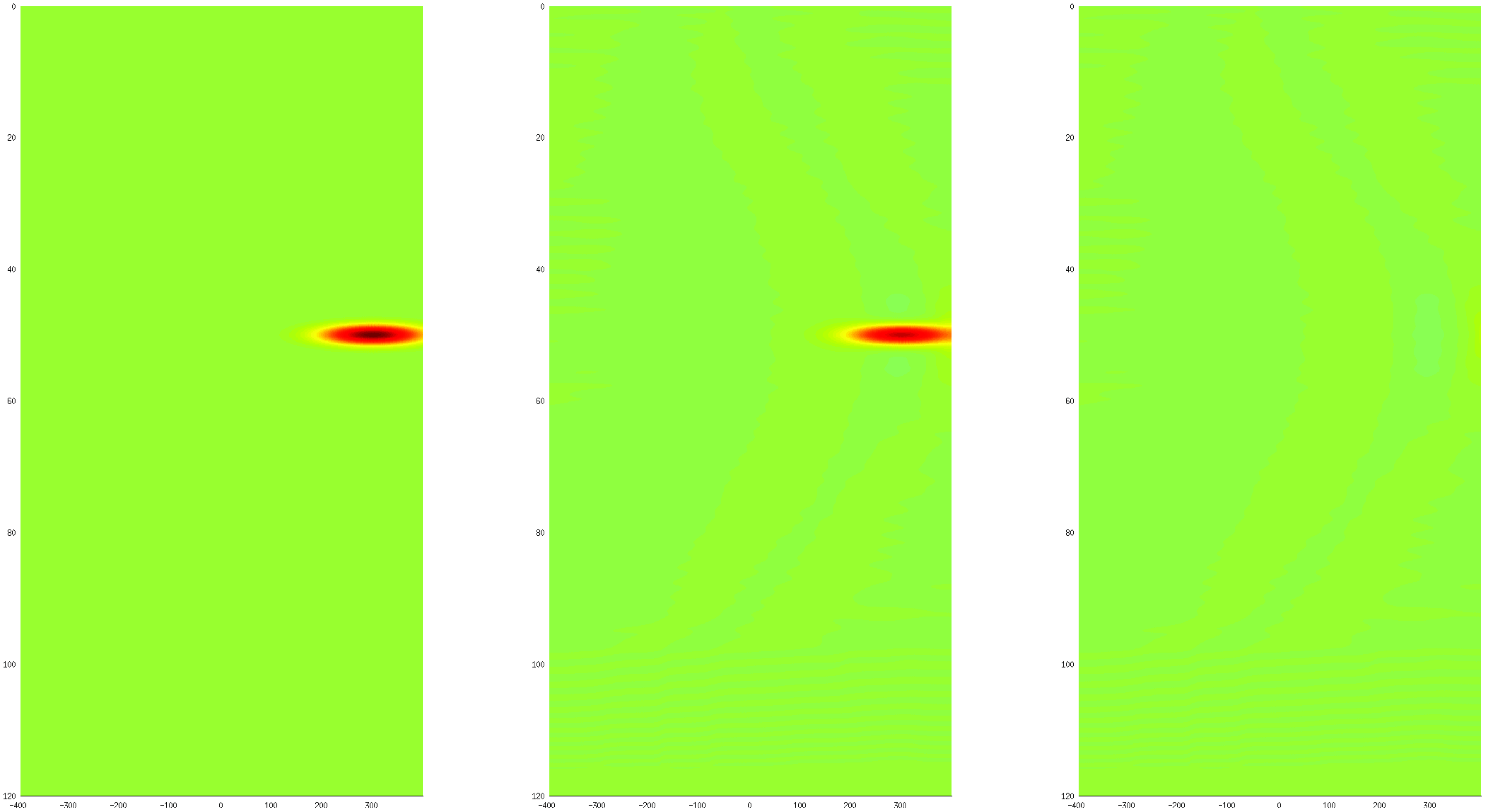
Kernel comparison



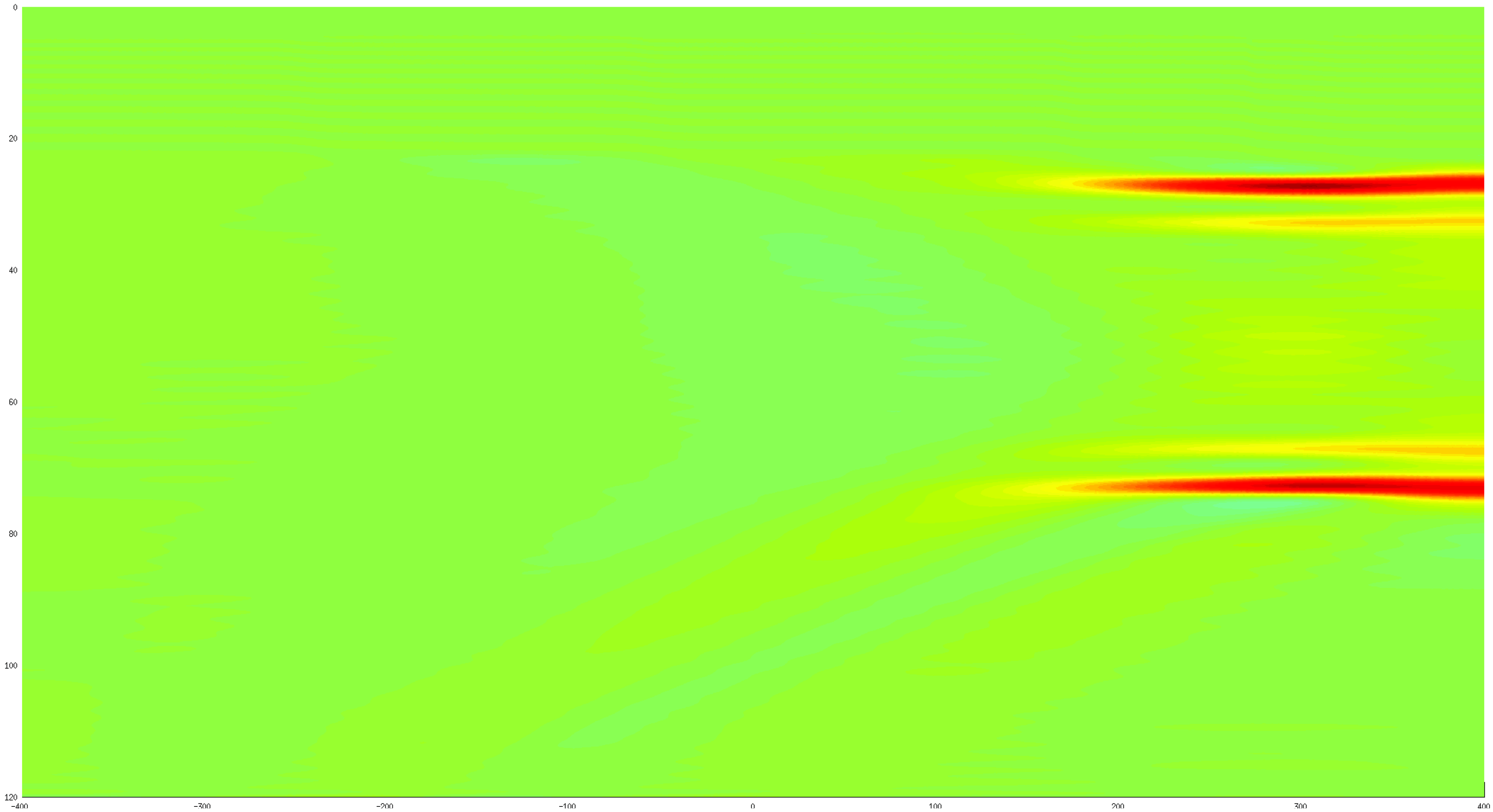
Optimal data functional



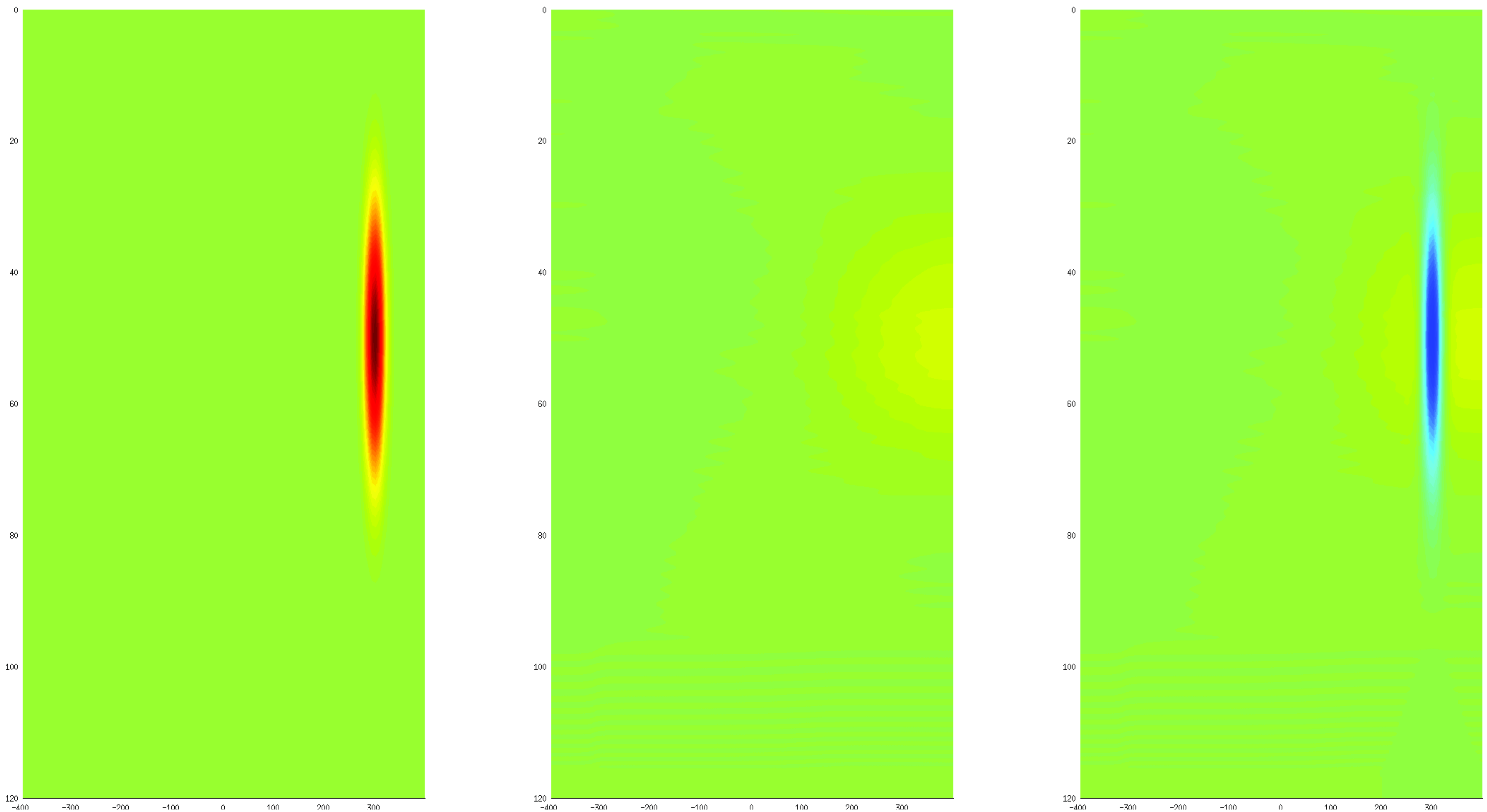
Kernel comparison



Optimal data functional



Kernel comparison



Summary

Convergence of gradient-based methods depends on starting model if problem is under-determined and not regularized

Backus-Gilbert theory provides a useful alternative to data-fitting methods

By using adjoint methods, Backus-Gilbert theory can be implemented iteratively in problems with large data-sets

Some of these ideas extend to non-linear problems ...